

Numerical Methods of Chaos Detection

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Outline

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 - ✓ **Poincaré Surface of Section**
 - ✓ **Lyapunov exponents**
- **Smaller ALignment Index – SALI**
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- **Generalized ALignment Index – GALI**
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 - ✓ **Global dynamics**
 - ✓ **Motion on low-dimensional tori**
- **Conclusions**

Autonomous Hamiltonian systems

Consider an **N degree of freedom** autonomous Hamiltonian system having a Hamiltonian function of the form:

$$H(\overbrace{q_1, q_2, \dots, q_N}^{\text{positions}}, \overbrace{p_1, p_2, \dots, p_N}^{\text{momenta}})$$

The time evolution of an orbit (trajectory) with initial condition

$$P(0) = (q_1(0), q_2(0), \dots, q_N(0), p_1(0), p_2(0), \dots, p_N(0))$$

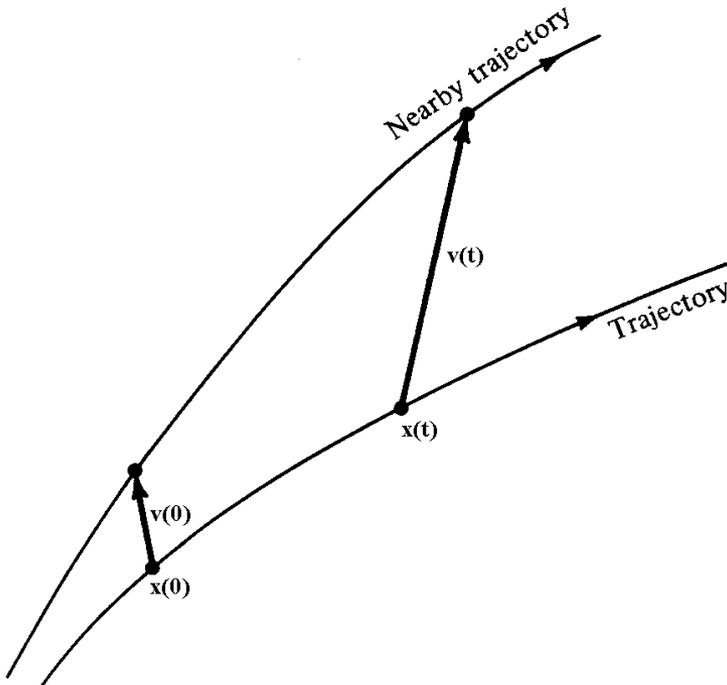
is governed by the **Hamilton's equations of motion**

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}$$

Variational Equations

We use the notation $\mathbf{x} = (q_1, q_2, \dots, q_N, p_1, p_2, \dots, p_N)^T$. The **deviation vector** from a given orbit is denoted by

$$\mathbf{v} = (\delta x_1, \delta x_2, \dots, \delta x_n)^T, \text{ with } n=2N$$



The time evolution of \mathbf{v} is given by the so-called **variational equations**:

$$\frac{d\mathbf{v}}{dt} = -\mathbf{J} \cdot \mathbf{P} \cdot \mathbf{v}$$

where

$$\mathbf{J} = \begin{pmatrix} \mathbf{0}_N & -\mathbf{I}_N \\ \mathbf{I}_N & \mathbf{0}_N \end{pmatrix}, \quad \mathbf{P}_{ij} = \frac{\partial^2 \mathbf{H}}{\partial \mathbf{x}_i \partial \mathbf{x}_j} \quad i, j = 1, 2, \dots, n$$

Example (Hénon-Heiles system)

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3$$

Hamilton's equations of motion:

$$\frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \quad \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \Rightarrow \begin{cases} \dot{x} = p_x \\ \dot{y} = p_y \\ \dot{p}_x = -x - 2xy \\ \dot{p}_y = -y - x^2 + y^2 \end{cases}$$

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In order to get the variational equations we **linearize** the above equations by substituting x, y, p_x, p_y with $x+v_1, y+v_2, p_x+v_3, p_y+v_4$ where $v=(v_1, v_2, v_3, v_4)$ is the deviation vector. So we get:

$$\dot{p}_x + \dot{v}_3 = -x - v_1 - 2(x + v_1)(y + v_2) \Rightarrow$$

$$\dot{p}_x + \dot{v}_3 = -x - v_1 - 2xy - 2xv_2 - 2yv_1 - 2v_1v_2 \Rightarrow$$

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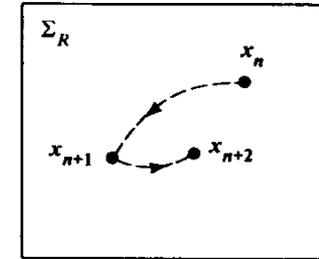
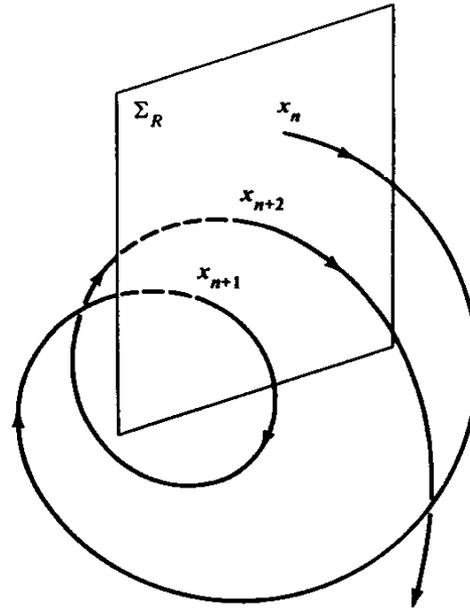
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$\dot{v}_1 = v_3$	+	$\dot{x} = p_x$
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$\dot{v}_4 = -v_2 - 2xv_1 + 2yv_2$		$\dot{p}_y = -y - x^2 + y^2$

Complete set of equations

Poincaré Surface of Section (PSS)

We can constrain the study of an $N+1$ degree of freedom Hamiltonian system to a **2N-dimensional subspace of the general phase space.**

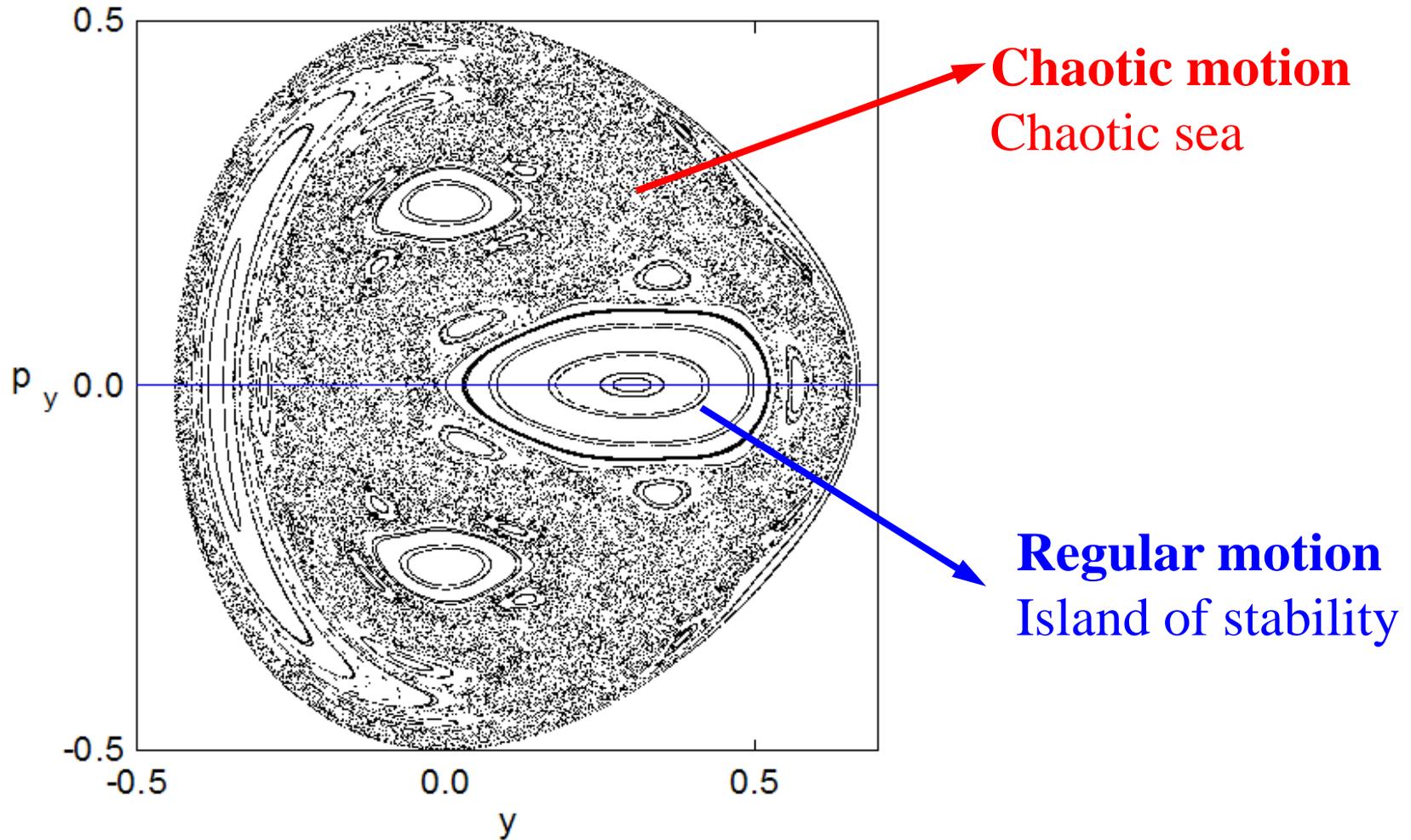


Lieberman & Lichtenberg, 1992, *Regular and Chaotic Dynamics*, Springer.

In general we can assume a PSS of the form $q_{N+1} = \text{constant}$. Then only variables $q_1, q_2, \dots, q_N, p_1, p_2, \dots, p_N$ are needed to describe the evolution of an orbit on the PSS, since p_{N+1} can be found from the Hamiltonian.

In this sense **an $N+1$ degree of freedom Hamiltonian system corresponds to a 2N-dimensional symplectic map.**

Hénon-Heiles system: PSS



Symplectic Maps

Consider an **2N-dimensional symplectic map T**. In this case we have **discrete time**.

This is an area-preserving map whose Jacobian matrix

$$\mathbf{M} = \frac{\partial \mathbf{T}}{\partial \mathbf{x}} = \begin{bmatrix} \frac{\partial \mathbf{T}_1}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{T}_1}{\partial \mathbf{x}_2} & \dots & \frac{\partial \mathbf{T}_1}{\partial \mathbf{x}_{2N}} \\ \frac{\partial \mathbf{T}_2}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{T}_2}{\partial \mathbf{x}_2} & \dots & \frac{\partial \mathbf{T}_2}{\partial \mathbf{x}_{2N}} \\ \vdots & \vdots & & \vdots \\ \frac{\partial \mathbf{T}_{2N}}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{T}_{2N}}{\partial \mathbf{x}_2} & \dots & \frac{\partial \mathbf{T}_{2N}}{\partial \mathbf{x}_{2N}} \end{bmatrix}$$

satisfies

$$\mathbf{M}^T \cdot \mathbf{J}_{2N} \cdot \mathbf{M} = \mathbf{J}_{2N}$$

Symplectic Maps

The evolution of an **orbit** with initial condition

$$\mathbf{P}(0) = (\mathbf{x}_1(0), \mathbf{x}_2(0), \dots, \mathbf{x}_{2N}(0))$$

is governed by the **equations of map T**

$$\mathbf{P}(i+1) = \mathbf{T} \mathbf{P}(i) \quad , \quad i = 0, 1, 2, \dots$$

The evolution of an initial **deviation vector**

$$\mathbf{v}(0) = (\delta \mathbf{x}_1(0), \delta \mathbf{x}_2(0), \dots, \delta \mathbf{x}_{2N}(0))$$

is given by the corresponding **tangent map**

$$\mathbf{v}(i+1) = \left. \frac{\partial \mathbf{T}}{\partial \mathbf{P}} \right|_i \cdot \mathbf{v}(i) \quad , \quad i = 0, 1, 2, \dots$$

Example – 2D map

Equations of the map:

$$\begin{pmatrix} \mathbf{x}'_1 \\ \mathbf{x}'_2 \end{pmatrix} = \mathbf{T} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} \Rightarrow \begin{aligned} \mathbf{x}'_1 &= \mathbf{x}_1 + \mathbf{x}_2 \\ \mathbf{x}'_2 &= \mathbf{x}_2 - v \sin(\mathbf{x}_1 + \mathbf{x}_2) \end{aligned} \quad (\text{mod } 2\pi)$$

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Tangent map:

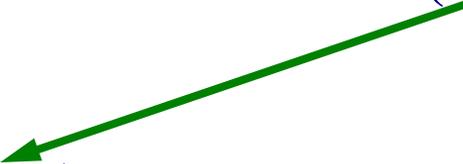
$$\mathbf{v}(\mathbf{i} + \mathbf{1}) = \left. \frac{\partial \mathbf{T}}{\partial \mathbf{P}} \right|_{\mathbf{i}} \cdot \mathbf{v}(\mathbf{i})$$

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$$\mathbf{v}(\mathbf{i} + 1) = \left. \frac{\partial \mathbf{T}}{\partial \mathbf{P}} \right|_{\mathbf{i}} \cdot \mathbf{v}(\mathbf{i})$$

$$\begin{pmatrix} d\mathbf{x}'_1 \\ d\mathbf{x}'_2 \end{pmatrix}$$

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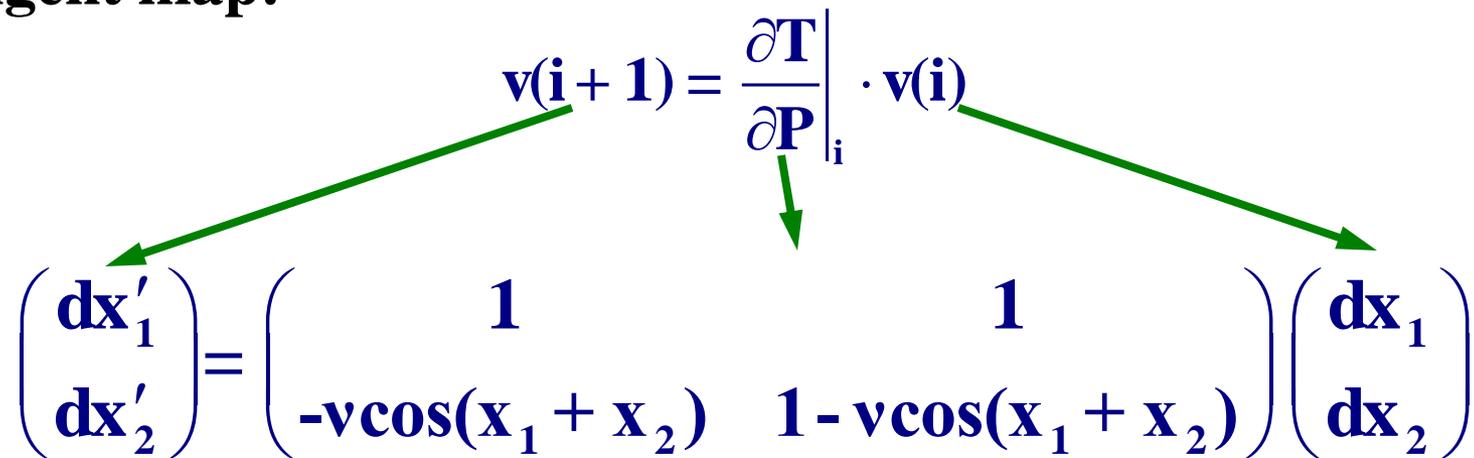
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Lyapunov Exponents

Roughly speaking, the Lyapunov exponents of a given orbit characterize the **mean exponential rate of divergence** of trajectories surrounding it.

Consider an orbit in the $2N$ -dimensional phase space with **initial condition $\mathbf{x}(0)$** and an **initial deviation vector from it $\mathbf{v}(0)$** . Then the mean exponential rate of divergence is:

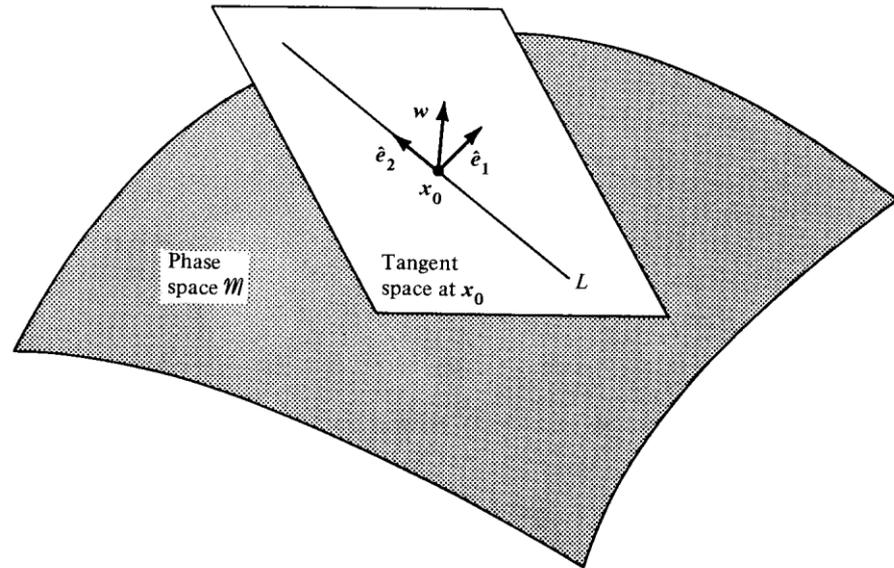
$$\sigma(\mathbf{x}(0), \mathbf{v}(0)) = \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{\|\mathbf{v}(t)\|}{\|\mathbf{v}(0)\|}$$

Lyapunov Exponents

There exists an **M-dimensional basis** $\{\hat{e}_i\}$ of v such that for any v , σ takes one of the M (possibly nondistinct) values

$$\sigma_i(x(0)) = \sigma(x(0), \hat{e}_i)$$

which are the **Lyapunov exponents**.



Benettin & Galgani, 1979, in Laval and Gressillon (eds.), op cit, 93

In autonomous Hamiltonian systems the M exponents are ordered in **pairs of opposite sign numbers and two of them are 0.**

Computation of the Maximal Lyapunov Exponent

Due to the exponential growth of $v(t)$ (and of $d(t)=\|v(t)\|$) we **renormalize $v(t)$** from time to time.

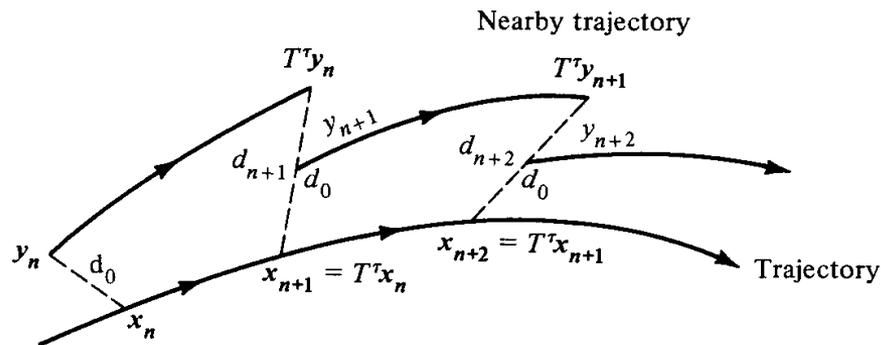


Figure 5.6. Numerical calculation of the maximal Liapunov characteristic exponent. Here $y = x + v$ and τ is a finite interval of time (after Benettin *et al.*, 1976).

Then the Maximal Lyapunov exponent is computed as

$$\sigma_1 = \lim_{n \rightarrow \infty} \frac{1}{n\tau} \sum_{i=1}^n \ln d_i$$

Maximum Lyapunov Exponent

$\sigma_1=0 \rightarrow$ Regular motion
 $\sigma_1 \neq 0 \rightarrow$ Chaotic motion

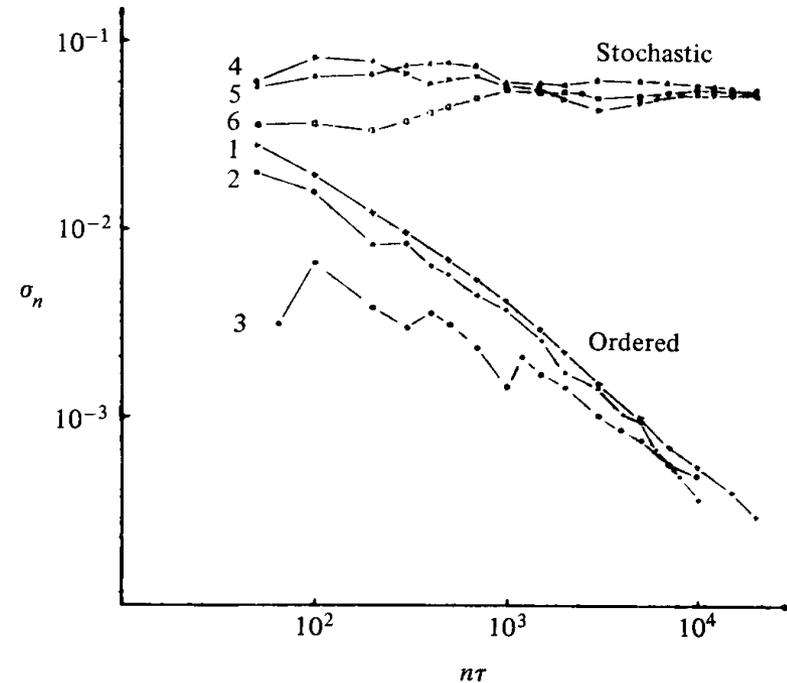


Figure 5.7. Behavior of σ_n at the intermediate energy $E = 0.125$ for initial points taken in the ordered (curves 1–3) or stochastic (curves 4–6) regions (after Benettin *et al.*, 1976).

If we start with more than one linearly independent deviation vectors they will **align to the direction defined by the largest Lyapunov exponent** for chaotic orbits.

**The
Smaller ALignment Index
(SALI)
method**

Definition of Smaller Alignment Index (SALI)

Consider the $2N$ -dimensional phase space of a conservative dynamical system (**symplectic map or Hamiltonian flow**).

An orbit in that space with initial condition :

$$P(0) = (x_1(0), x_2(0), \dots, x_{2N}(0))$$

and a deviation vector

$$v(0) = (\delta x_1(0), \delta x_2(0), \dots, \delta x_{2N}(0))$$

The evolution in time (in maps the time is discrete and is equal to the number n of the iterations) of a deviation vector is defined by:

- the **variational equations** (for Hamiltonian flows) and
- the equations of the **tangent map** (for mappings)

Definition of SALI

We follow the evolution in time of two different initial deviation vectors ($\mathbf{v}_1(\mathbf{0})$, $\mathbf{v}_2(\mathbf{0})$), and define SALI (**Ch.S. 2001, J. Phys. A**) as:

$$\text{SALI}(\mathbf{t}) = \min \left\{ \left\| \hat{\mathbf{v}}_1(\mathbf{t}) + \hat{\mathbf{v}}_2(\mathbf{t}) \right\|, \left\| \hat{\mathbf{v}}_1(\mathbf{t}) - \hat{\mathbf{v}}_2(\mathbf{t}) \right\| \right\}$$

where

$$\hat{\mathbf{v}}_1(\mathbf{t}) = \frac{\mathbf{v}_1(\mathbf{t})}{\|\mathbf{v}_1(\mathbf{t})\|}$$

When the two vectors become **collinear**

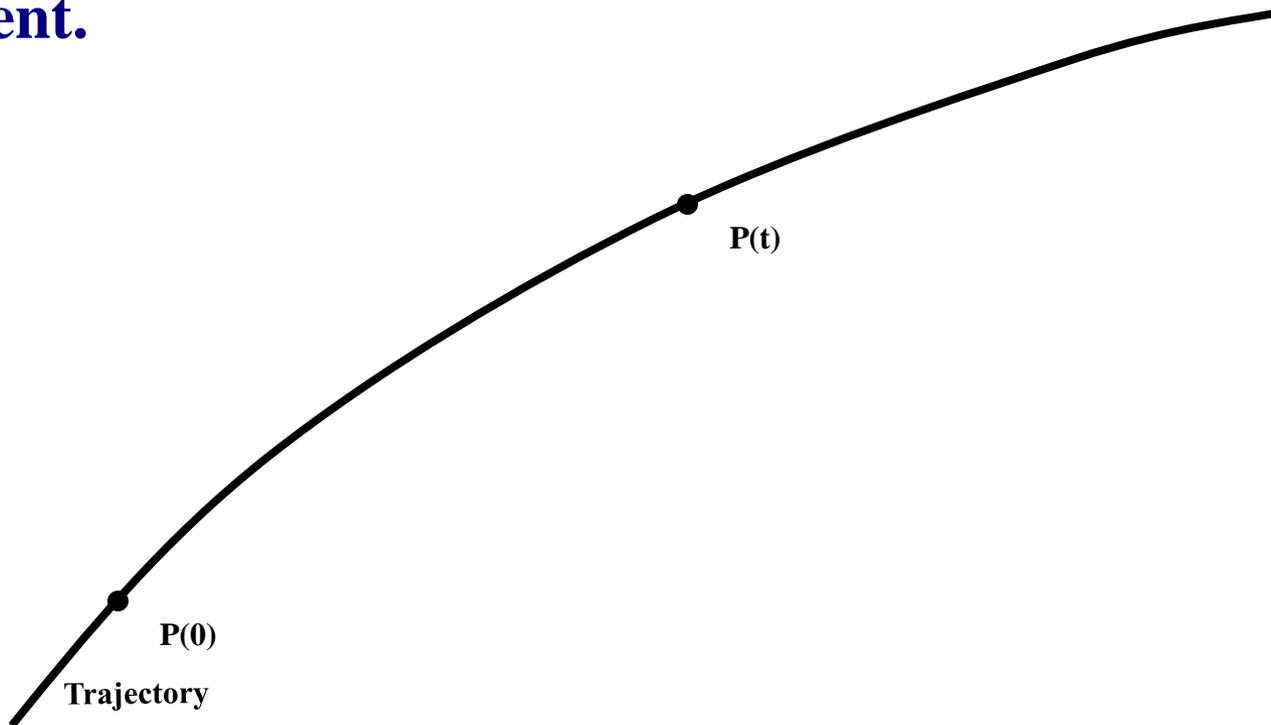
$$\text{SALI}(\mathbf{t}) \rightarrow \mathbf{0}$$

Behavior of SALI for chaotic motion

For chaotic orbits the two initially different deviation vectors tend to coincide with the direction defined by the maximum Lyapunov exponent.

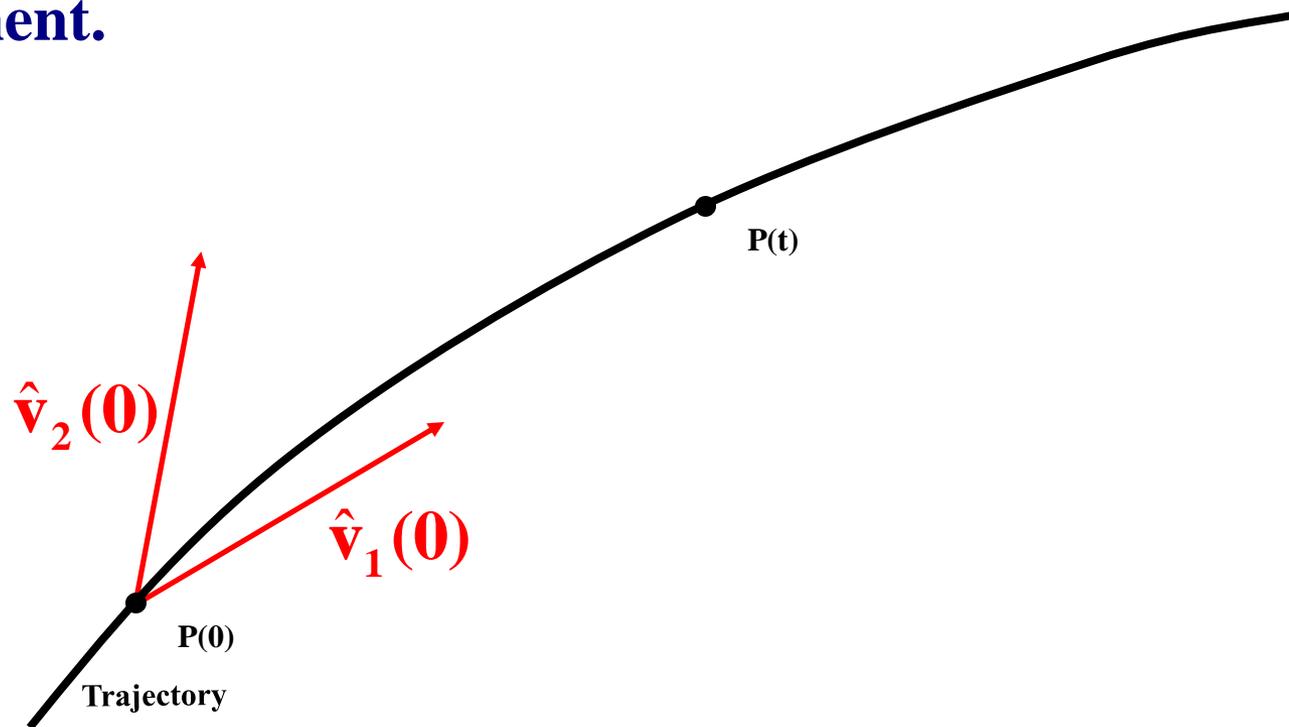
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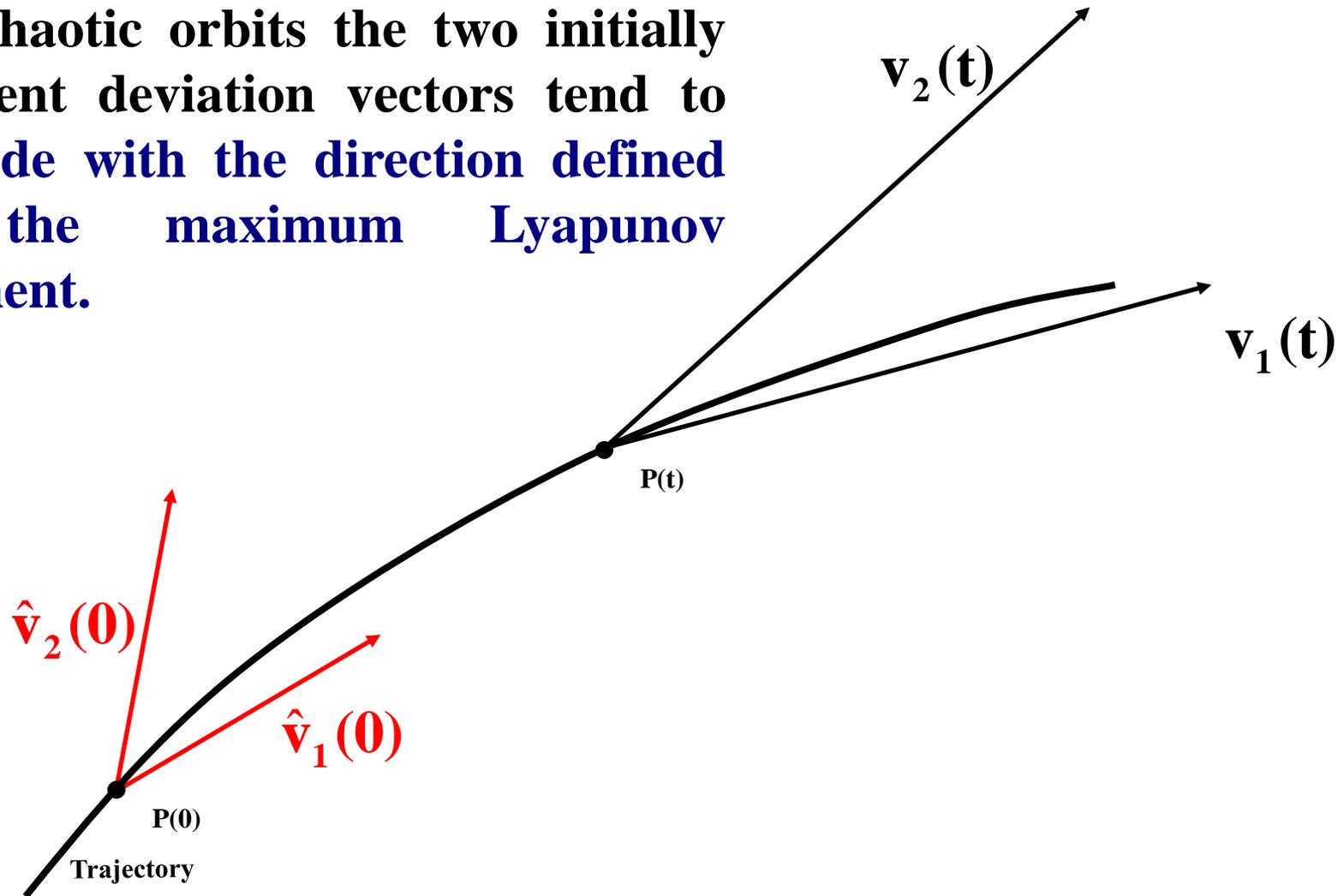
Behavior of SALI for chaotic motion

For chaotic orbits the two initially different deviation vectors tend to coincide with the direction defined by the maximum Lyapunov exponent.



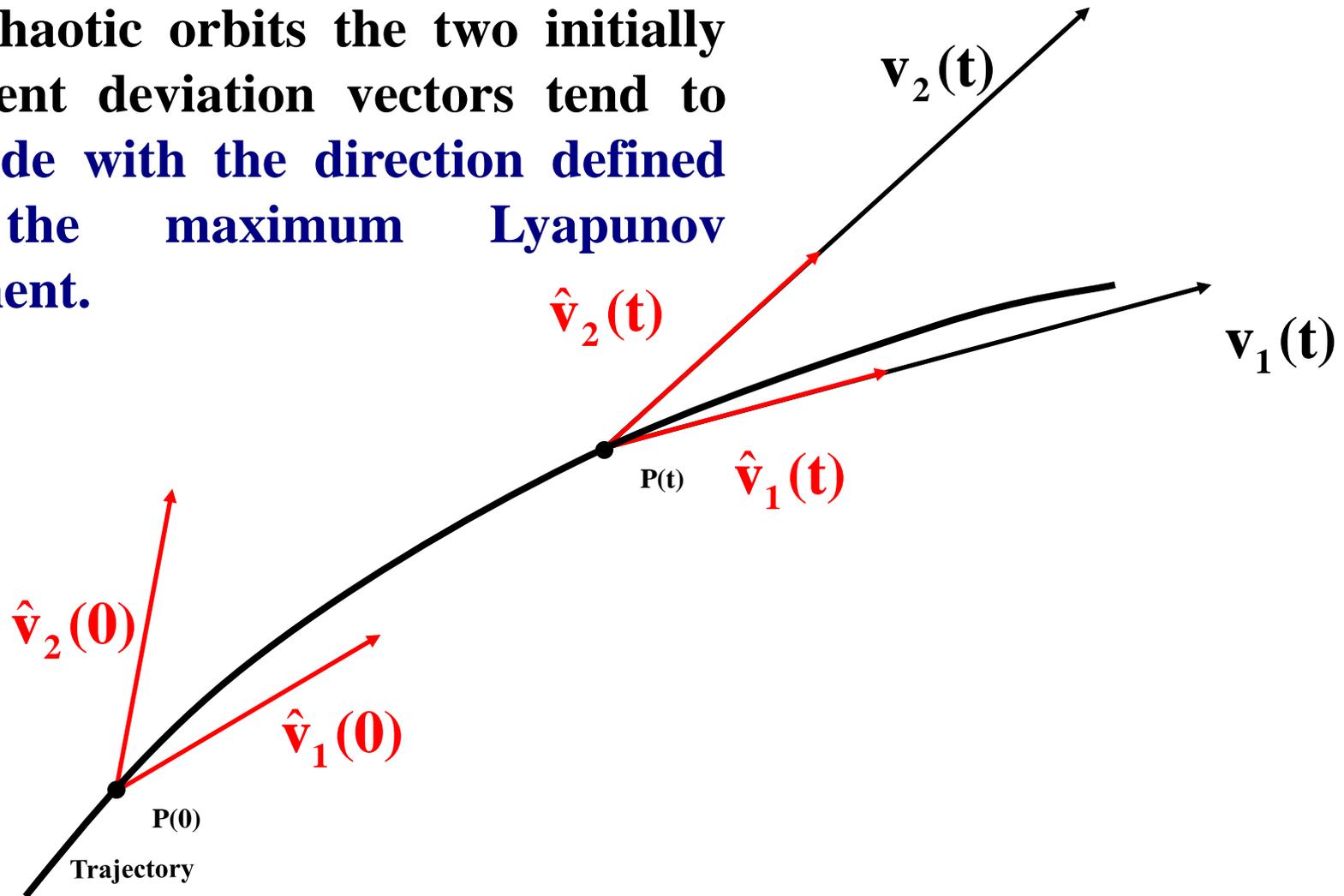
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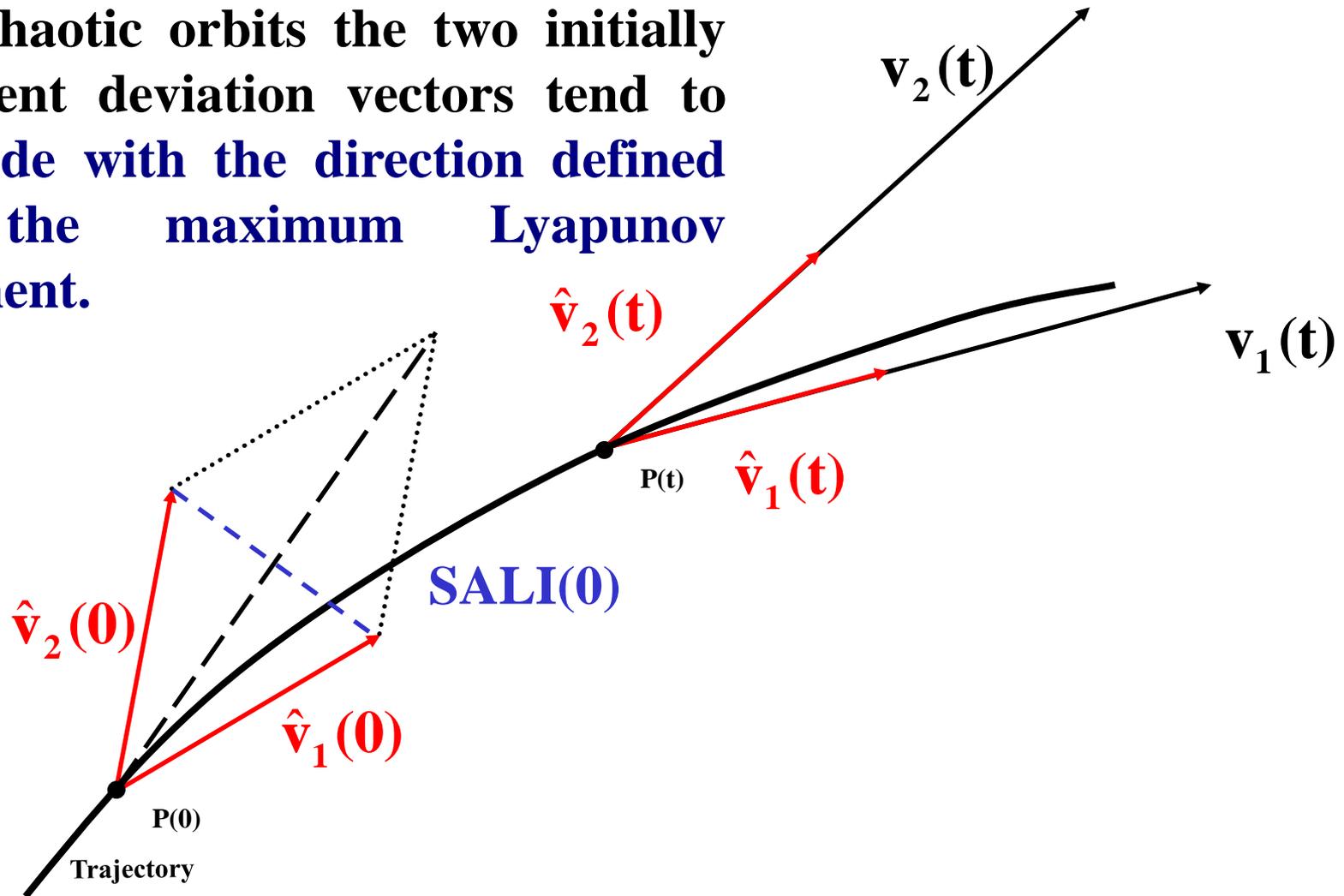
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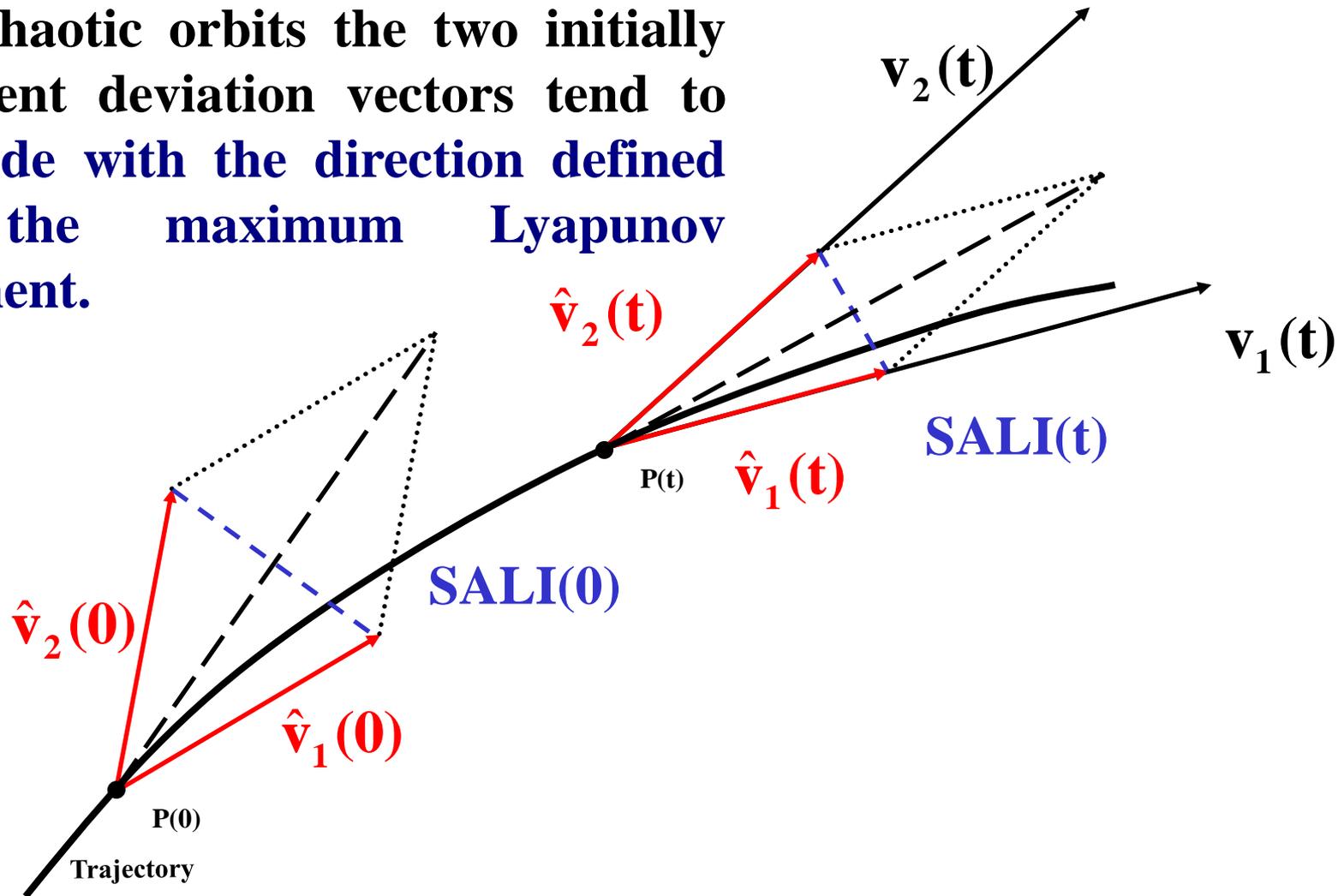
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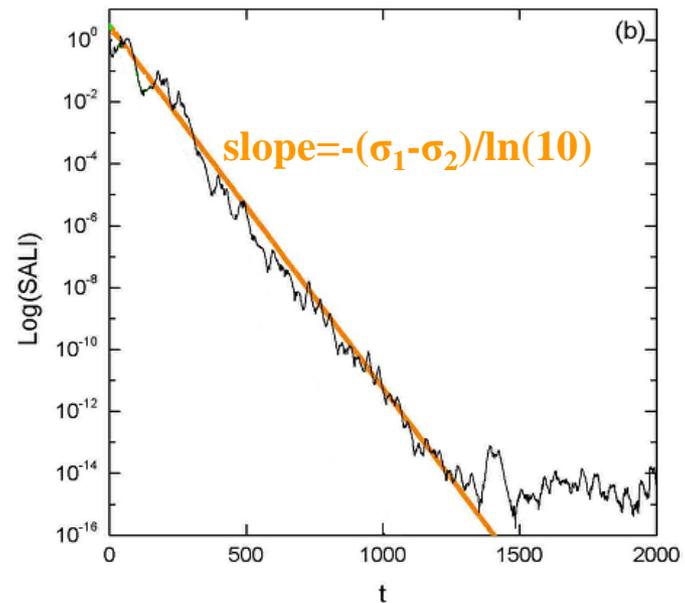
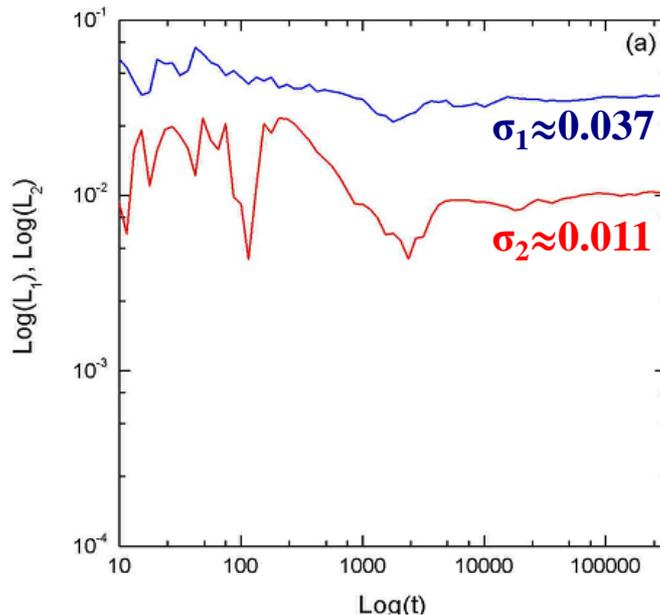


Behavior of SALI for chaotic motion

We test the validity of the approximation $\text{SALI} \propto e^{-(\sigma_1 - \sigma_2)t}$ (Ch.S., Antonopoulos, Bountis, Vrahatis, 2004, J. Phys. A) for a chaotic orbit of the 3D Hamiltonian

$$H = \sum_{i=1}^3 \frac{\omega_i}{2} (q_i^2 + p_i^2) + q_1^2 q_2 + q_1^2 q_3$$

with $\omega_1=1$, $\omega_2=1.4142$, $\omega_3=1.7321$, $H=0.09$

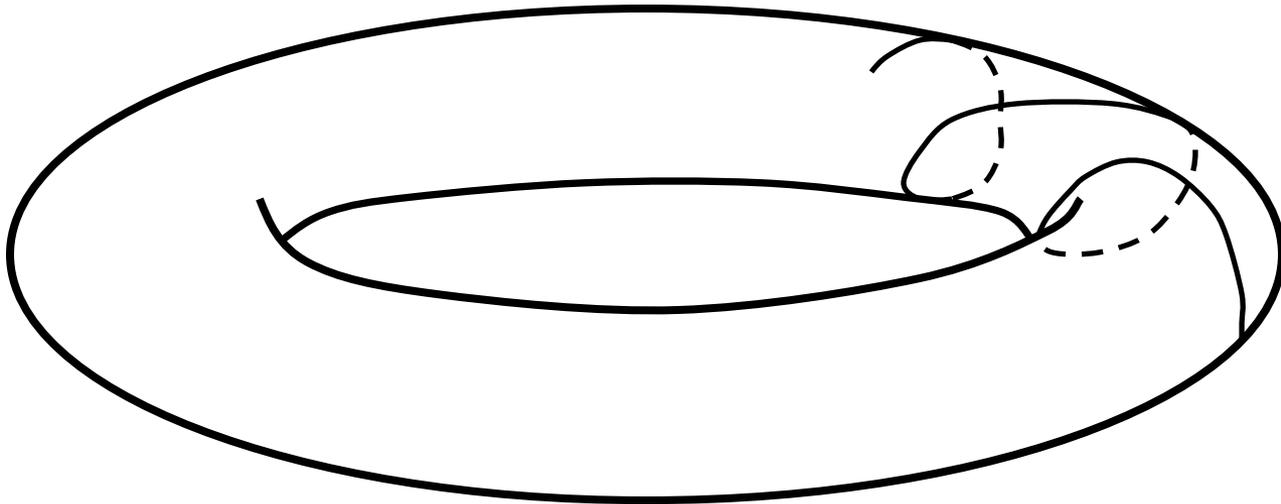


Behavior of SALI for regular motion

Regular motion occurs on a torus and two different initial deviation vectors become tangent to the torus, generally having different directions.

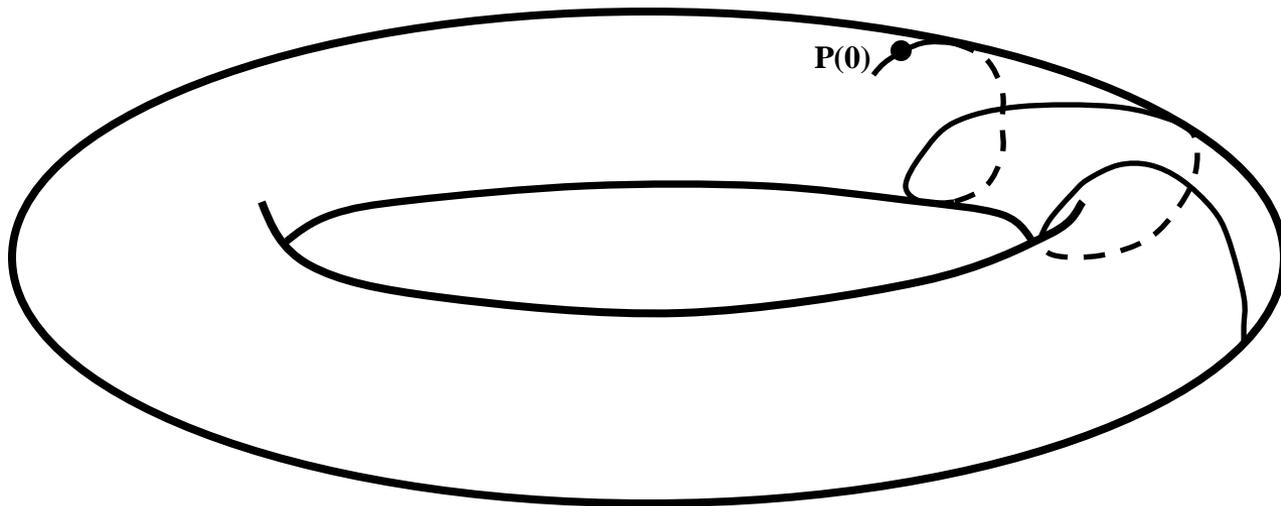
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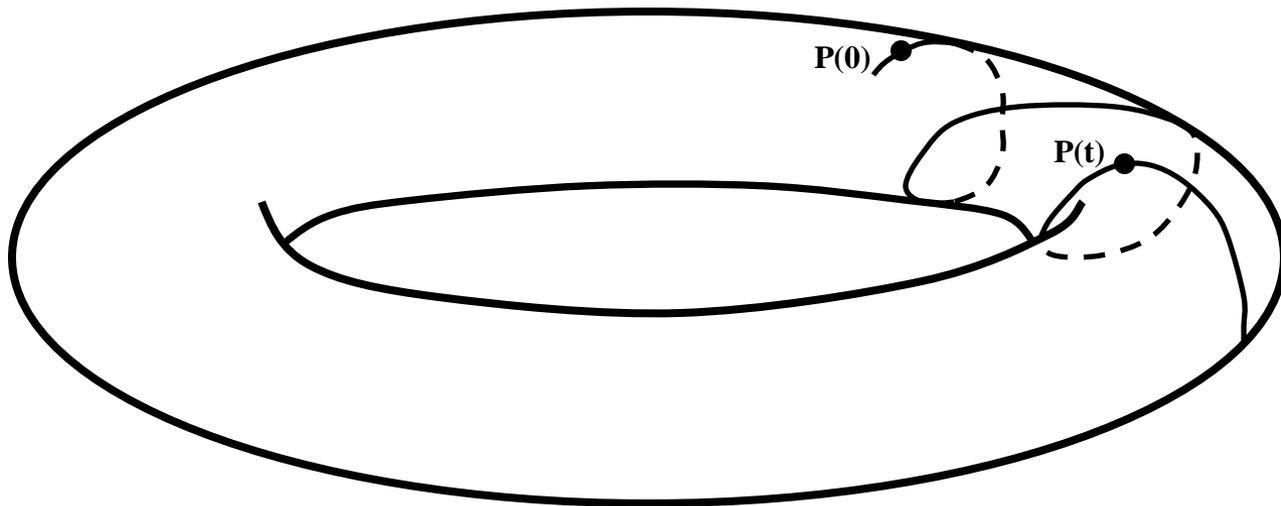
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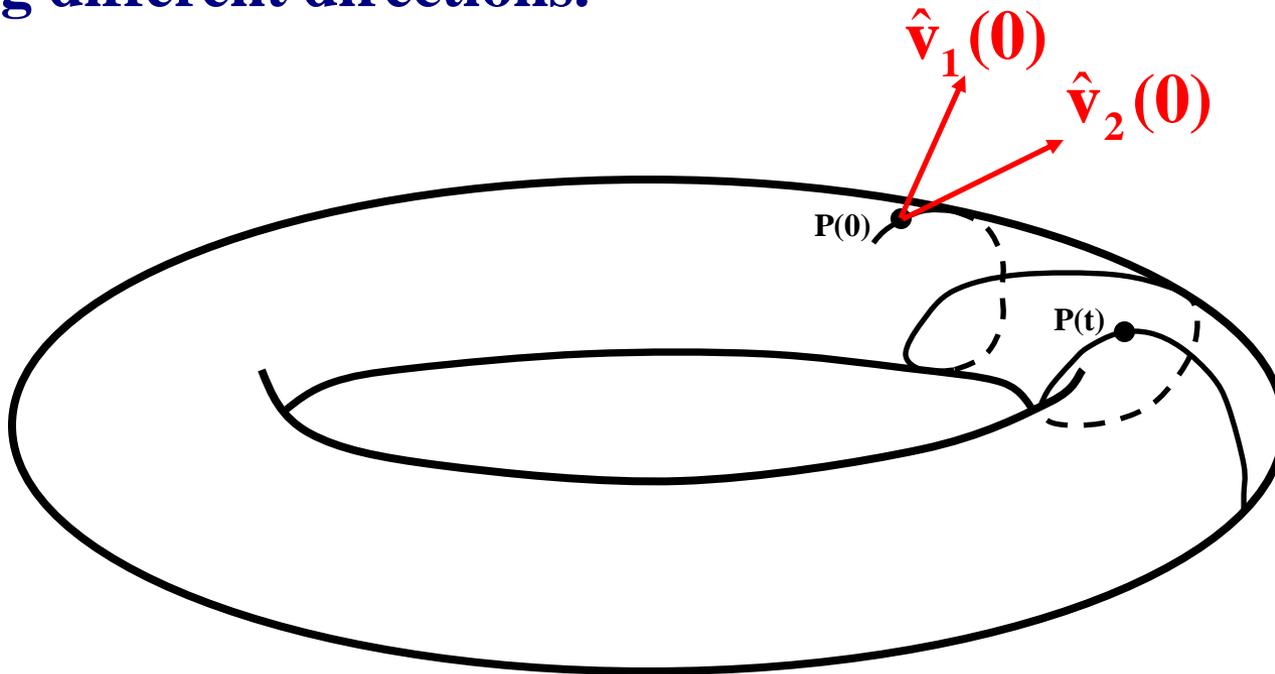
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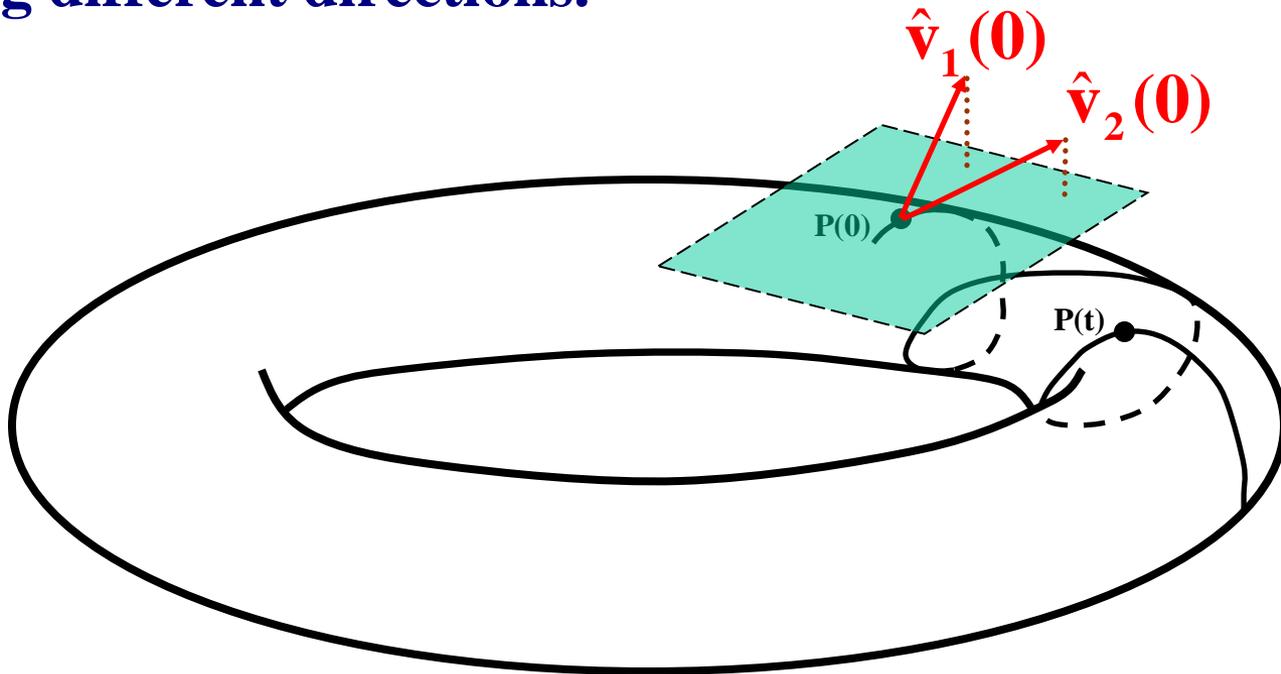
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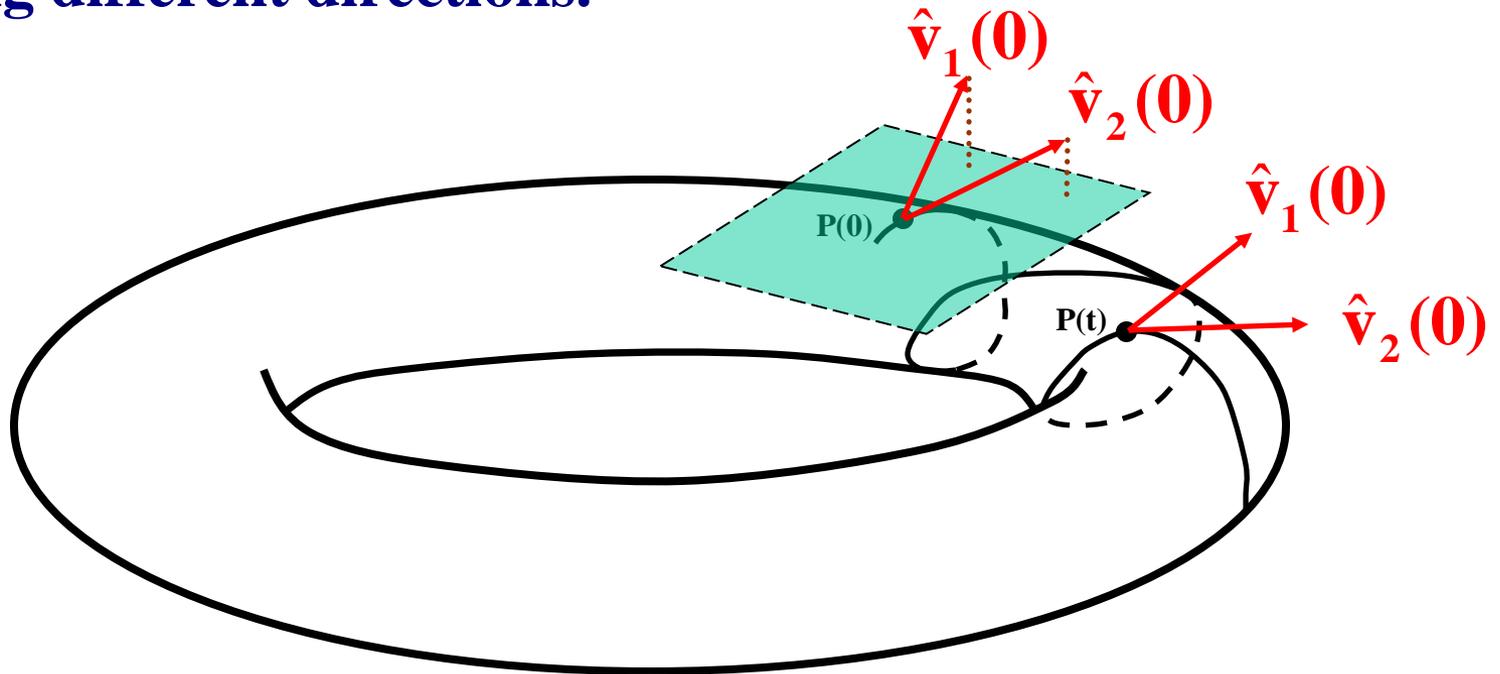
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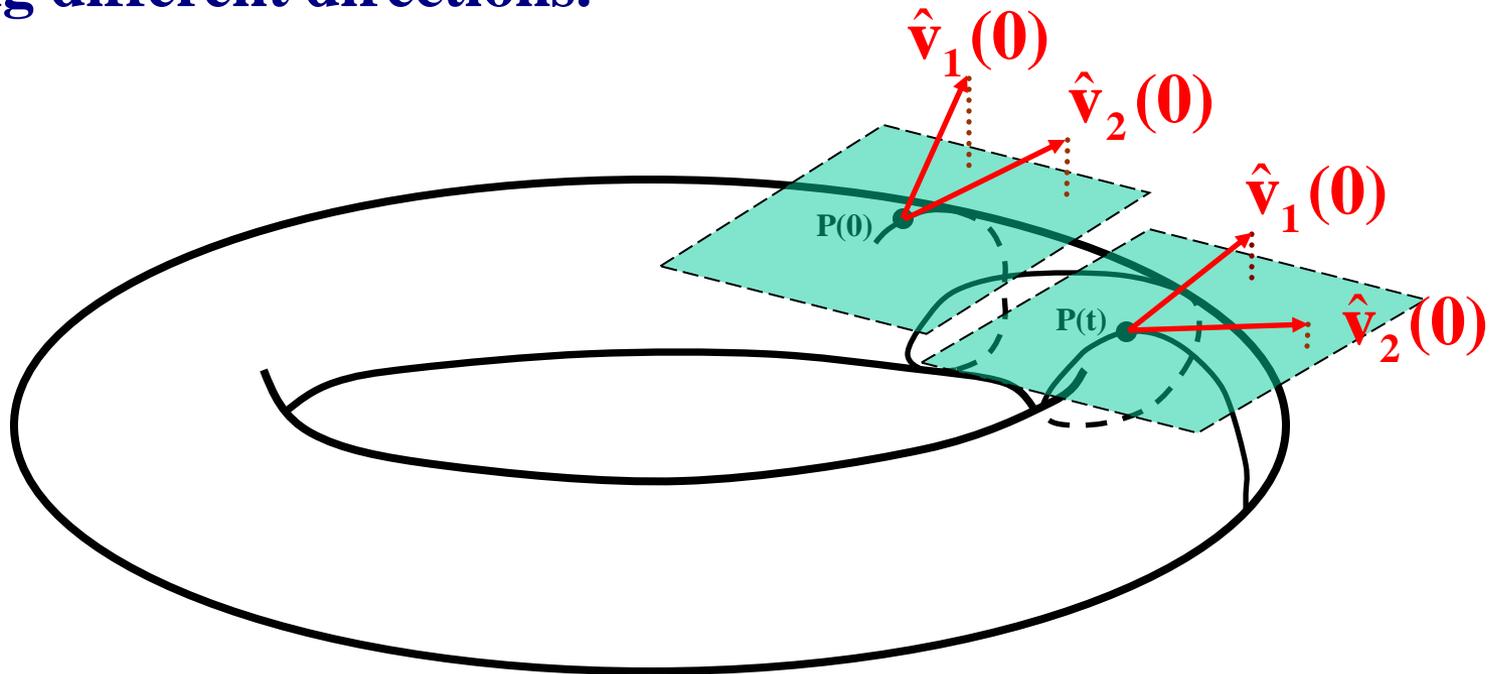
Behavior of SALI for regular motion

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Behavior of SALI for regular motion

Regular motion occurs on a torus and two different initial deviation vectors become tangent to the torus, generally having different directions.



Applications – Hénon-Heiles system

As an example, we consider the 2D Hénon-Heiles system:

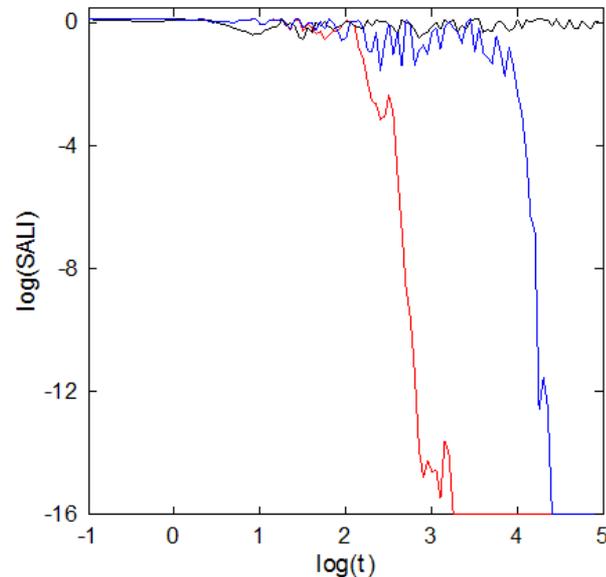
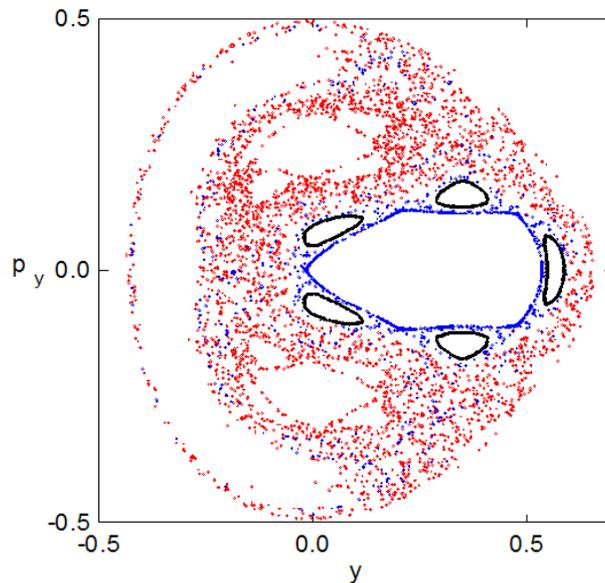
$$H_2 = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2y - \frac{1}{3}y^3$$

For $E=1/8$ we consider the orbits with initial conditions:

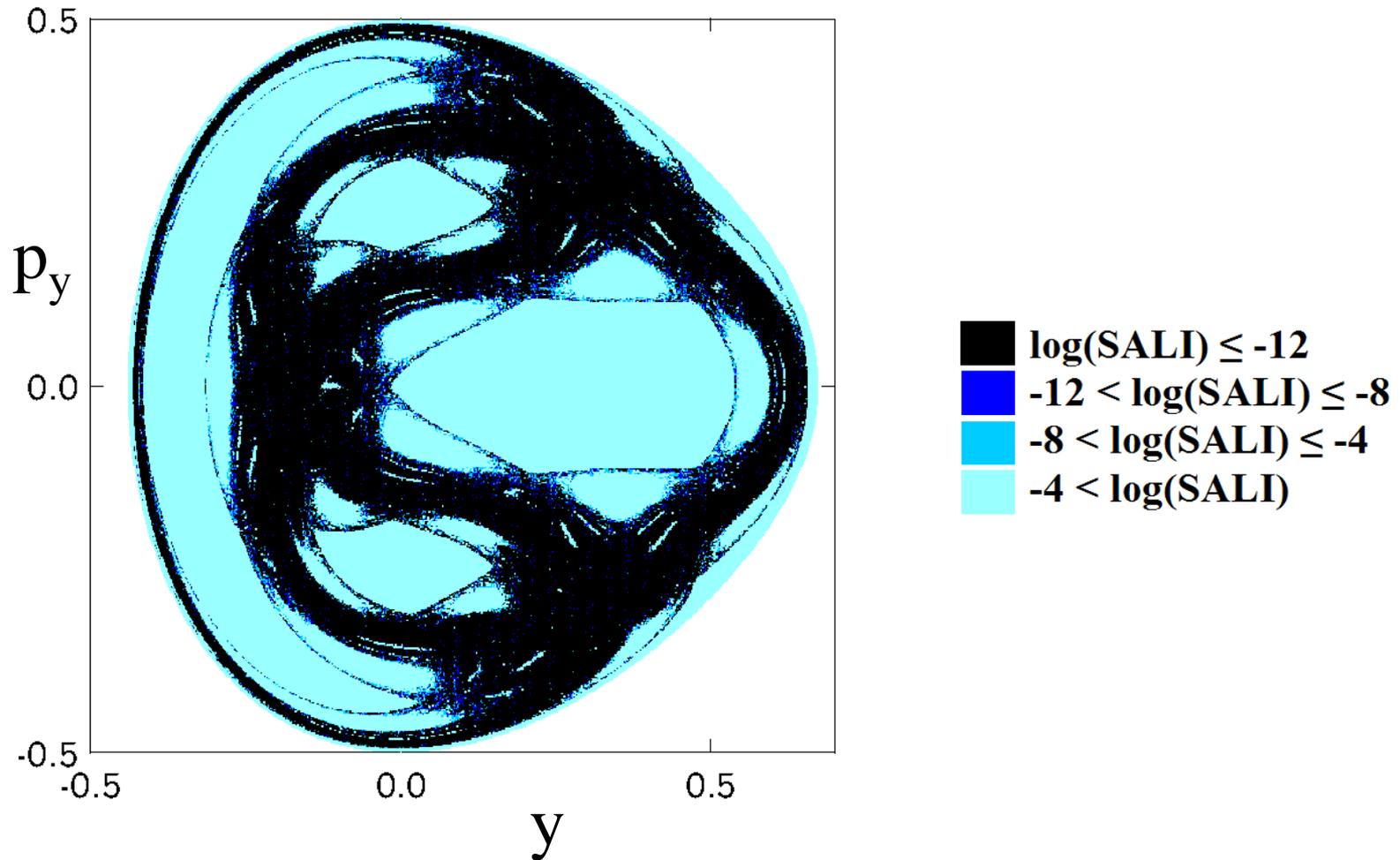
Regular orbit, $x=0$, $y=0.55$, $p_x=0.2417$, $p_y=0$

Chaotic orbit, $x=0$, $y=-0.016$, $p_x=0.49974$, $p_y=0$

Chaotic orbit, $x=0$, $y=-0.01344$, $p_x=0.49982$, $p_y=0$



Applications – Hénon-Heiles system



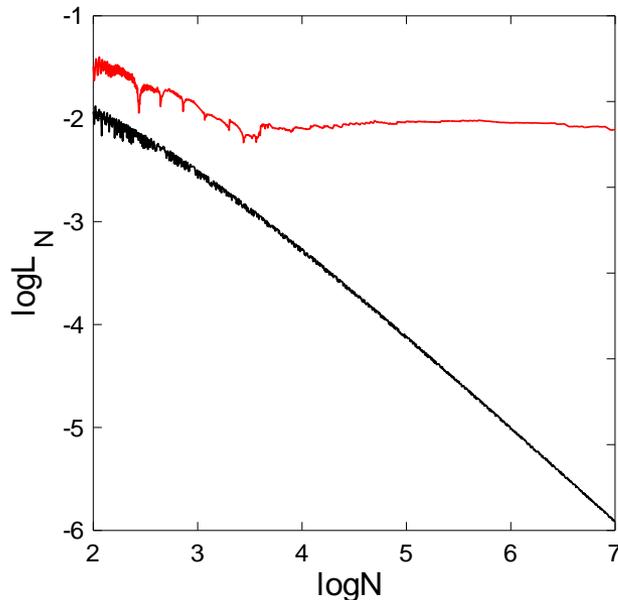
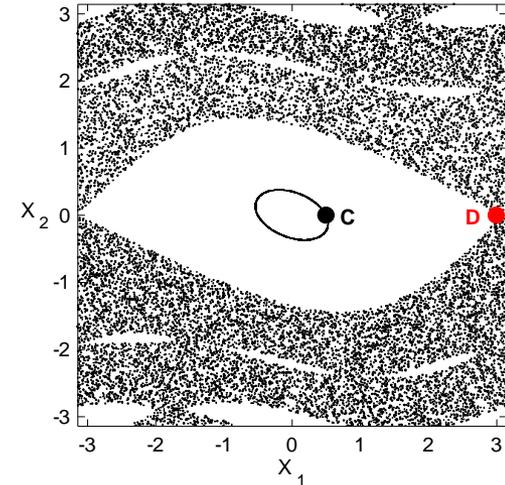
Applications – 4D map

$$\begin{aligned} \mathbf{x}'_1 &= \mathbf{x}_1 + \mathbf{x}_2 \\ \mathbf{x}'_2 &= \mathbf{x}_2 - \nu \sin(\mathbf{x}_1 + \mathbf{x}_2) - \mu [1 - \cos(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4)] \\ \mathbf{x}'_3 &= \mathbf{x}_3 + \mathbf{x}_4 \\ \mathbf{x}'_4 &= \mathbf{x}_4 - \kappa \sin(\mathbf{x}_3 + \mathbf{x}_4) - \mu [1 - \cos(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4)] \end{aligned} \quad (\text{mod } 2\pi)$$

For $\nu=0.5$, $\kappa=0.1$, $\mu=0.1$ we consider the orbits:

regular orbit C with initial conditions $x_1=0.5$, $x_2=0$, $x_3=0.5$, $x_4=0$.

chaotic orbit D with initial conditions $x_1=3$, $x_2=0$, $x_3=0.5$, $x_4=0$.



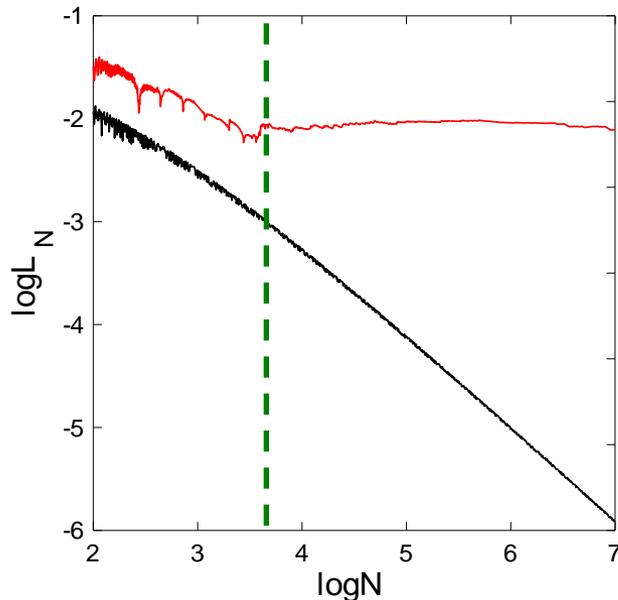
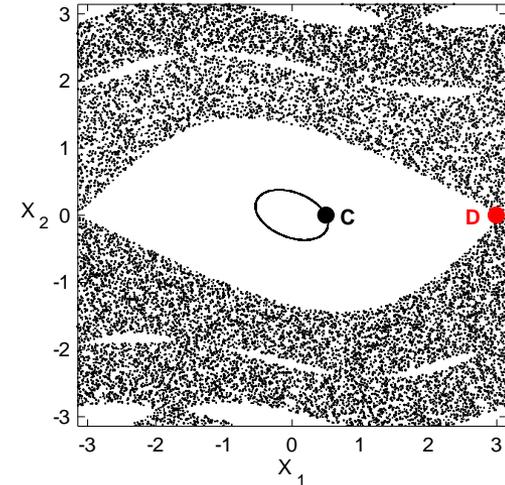
Applications – 4D map

$$\begin{aligned}
 \mathbf{x}'_1 &= \mathbf{x}_1 + \mathbf{x}_2 \\
 \mathbf{x}'_2 &= \mathbf{x}_2 - \nu \sin(\mathbf{x}_1 + \mathbf{x}_2) - \mu [1 - \cos(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4)] \\
 \mathbf{x}'_3 &= \mathbf{x}_3 + \mathbf{x}_4 \\
 \mathbf{x}'_4 &= \mathbf{x}_4 - \kappa \sin(\mathbf{x}_3 + \mathbf{x}_4) - \mu [1 - \cos(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4)]
 \end{aligned} \pmod{2\pi}$$

For $\nu=0.5$, $\kappa=0.1$, $\mu=0.1$ we consider the orbits:

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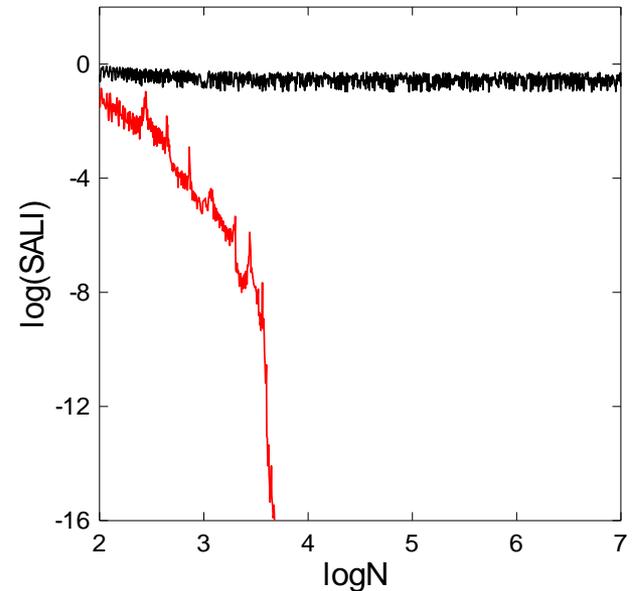
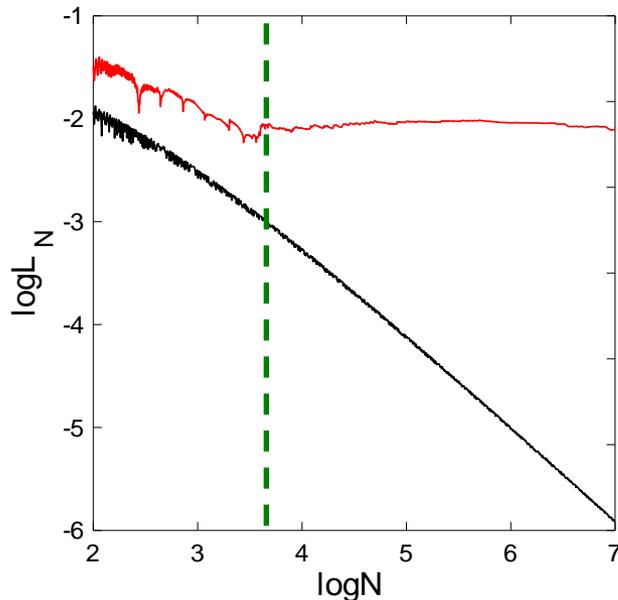
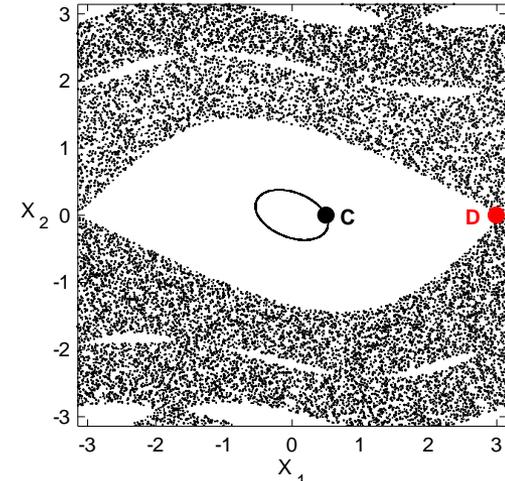
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 \mathbf{x}'_3 &= \mathbf{x}_3 + \mathbf{x}_4 \\
 \mathbf{x}'_4 &= \mathbf{x}_4 - \kappa \sin(\mathbf{x}_3 + \mathbf{x}_4) - \mu [1 - \cos(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4)]
 \end{aligned} \pmod{2\pi}$$

For $\nu=0.5$, $\kappa=0.1$, $\mu=0.1$ we consider the orbits:

regular orbit C with initial conditions $x_1=0.5$, $x_2=0$, $x_3=0.5$, $x_4=0$.

chaotic orbit D with initial conditions $x_1=3$, $x_2=0$, $x_3=0.5$, $x_4=0$.



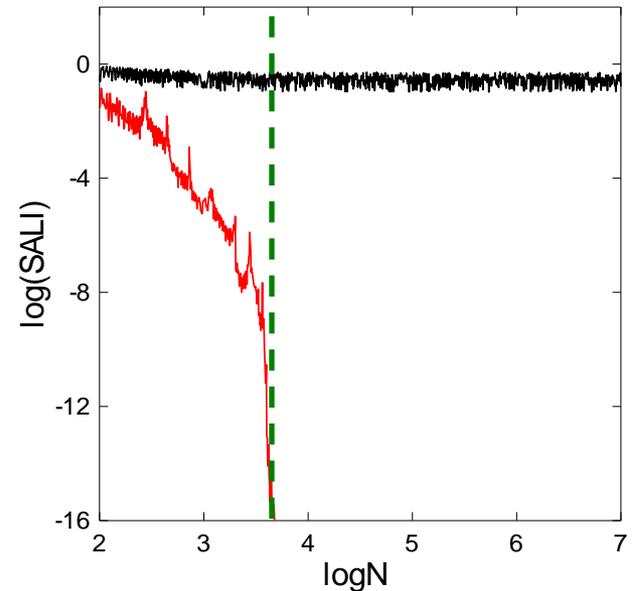
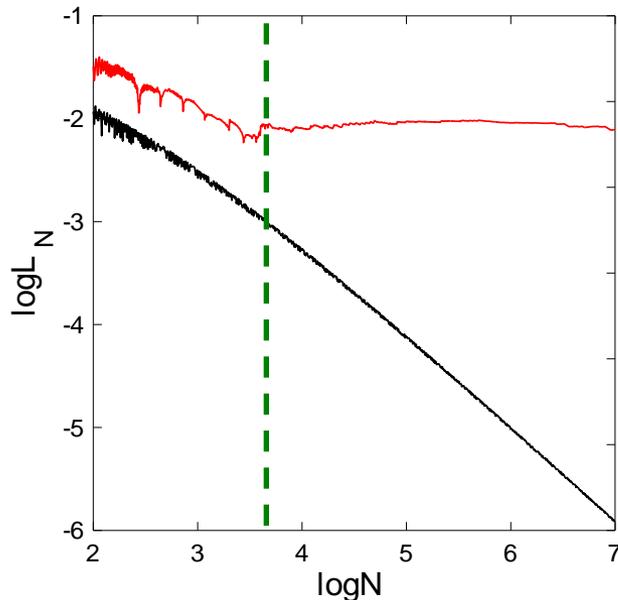
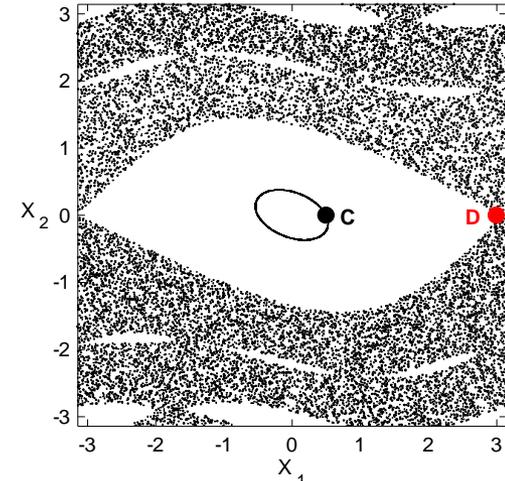
Applications – 4D map

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 \mathbf{x}'_3 &= \mathbf{x}_3 + \mathbf{x}_4 \\
 \mathbf{x}'_4 &= \mathbf{x}_4 - \kappa \sin(\mathbf{x}_3 + \mathbf{x}_4) - \mu [1 - \cos(\mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \mathbf{x}_4)]
 \end{aligned} \pmod{2\pi}$$

For $\nu=0.5$, $\kappa=0.1$, $\mu=0.1$ we consider the orbits:

regular orbit C with initial conditions $x_1=0.5, x_2=0, x_3=0.5, x_4=0$.

chaotic orbit D with initial conditions $x_1=3, x_2=0, x_3=0.5, x_4=0$.



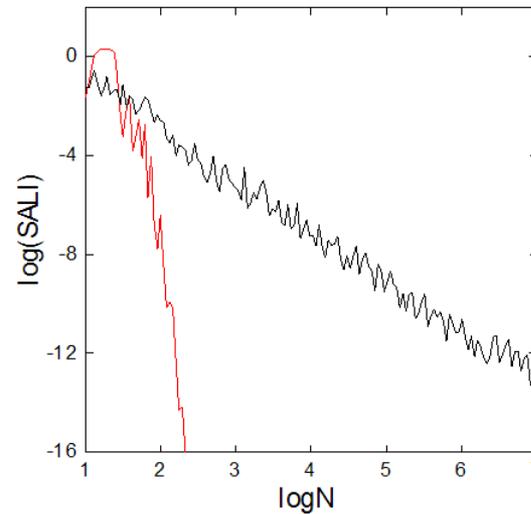
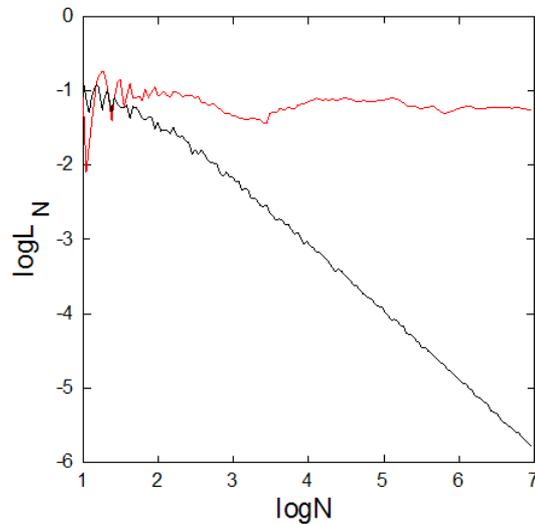
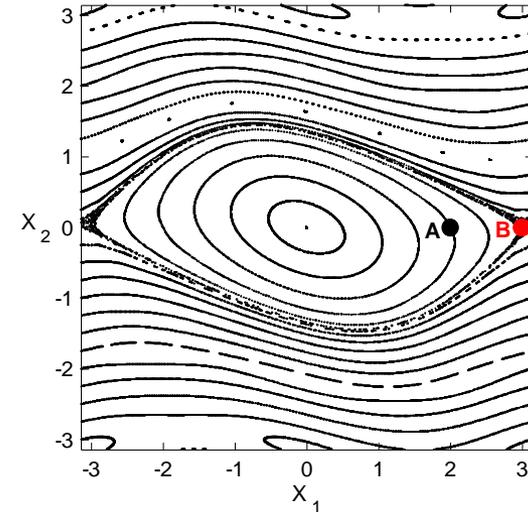
Applications – 2D map

$$\begin{aligned} \mathbf{x}'_1 &= \mathbf{x}_1 + \mathbf{x}_2 \\ \mathbf{x}'_2 &= \mathbf{x}_2 - \nu \sin(\mathbf{x}_1 + \mathbf{x}_2) \end{aligned} \quad (\text{mod } 2\pi)$$

For $\nu=0.5$ we consider the orbits:

regular orbit A with initial conditions $x_1=2, x_2=0$.

chaotic orbit B with initial conditions $x_1=3, x_2=0$.



Behavior of SALI

2D maps

SALI $\rightarrow 0$ both for regular and chaotic orbits

following, however, completely different time rates which allows us to distinguish between the two cases.

Hamiltonian flows and multidimensional maps

SALI $\rightarrow 0$ for chaotic orbits

SALI \rightarrow constant $\neq 0$ for regular orbits

Questions

Can we generalize SALI so that the new index:

- **Can rapidly reveal the nature of chaotic orbits with $\sigma_1 \approx \sigma_2$ ($\text{SALI} \propto e^{-(\sigma_1 - \sigma_2)t}$)?**
- **Depends on several Lyapunov exponents for chaotic orbits?**
- **Exhibits power-law decay for regular orbits depending on the dimensionality of the tangent space of the reference orbit as for 2D maps?**

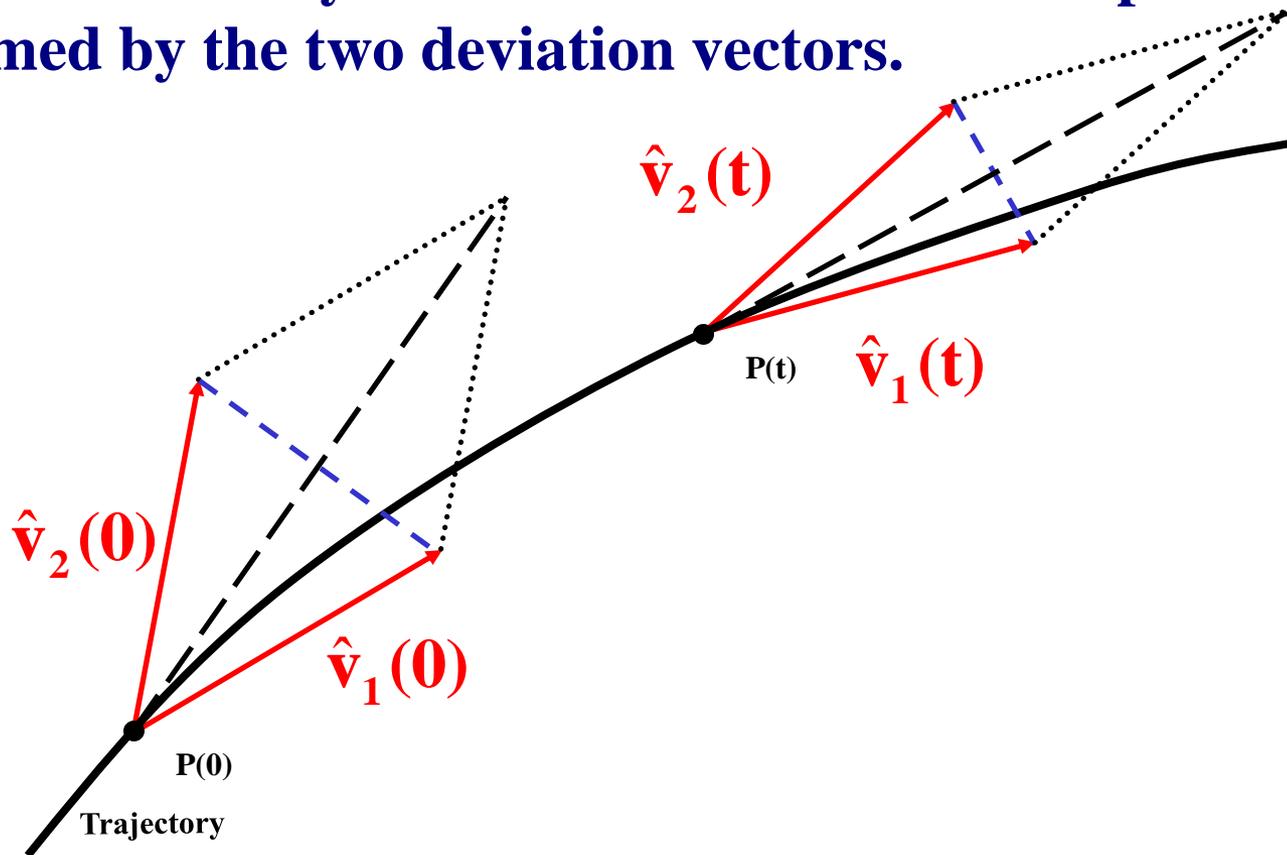
**The
Generalized ALignment Indices
(GALIs)
method**

Definition of Generalized Alignment Index (GALI)

SALI effectively measures the 'area' of the parallelogram formed by the two deviation vectors.

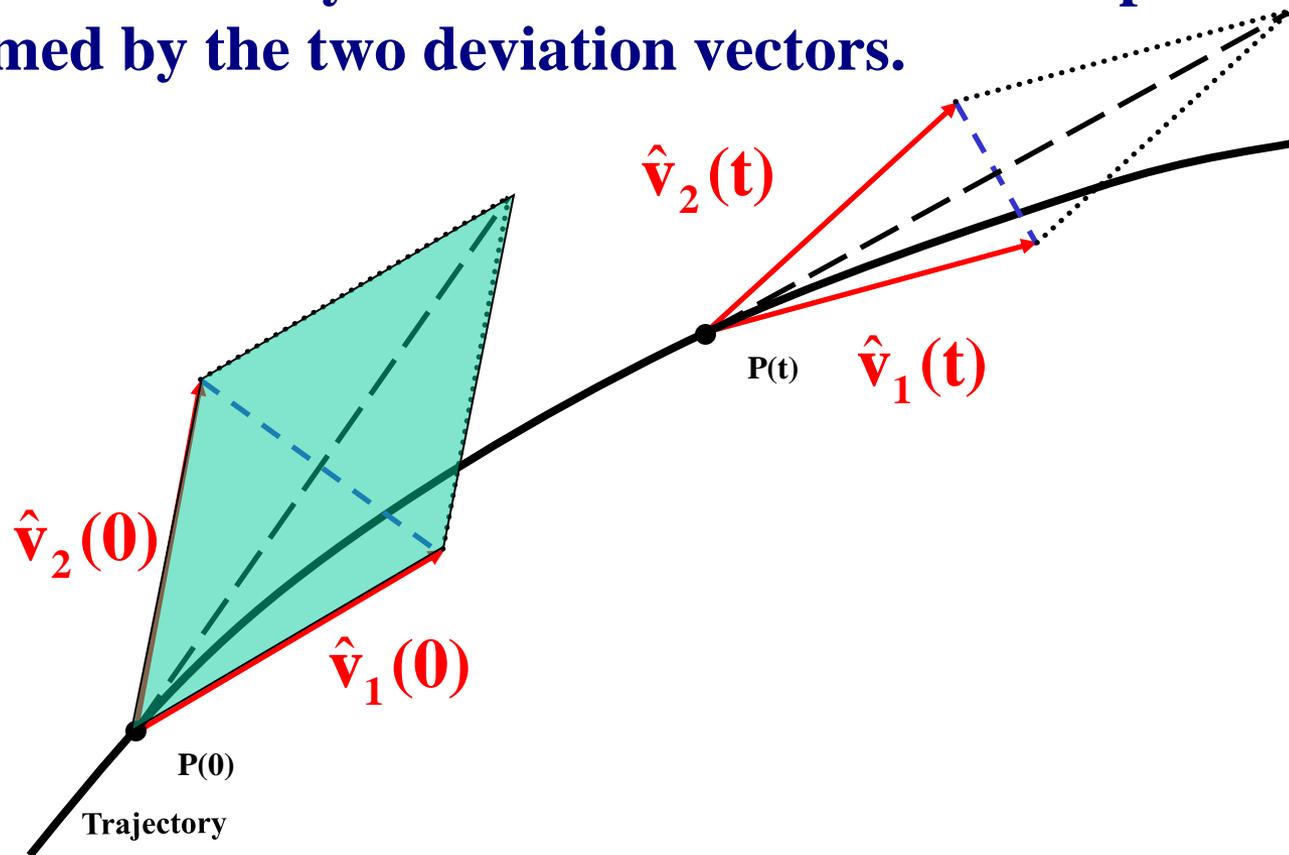
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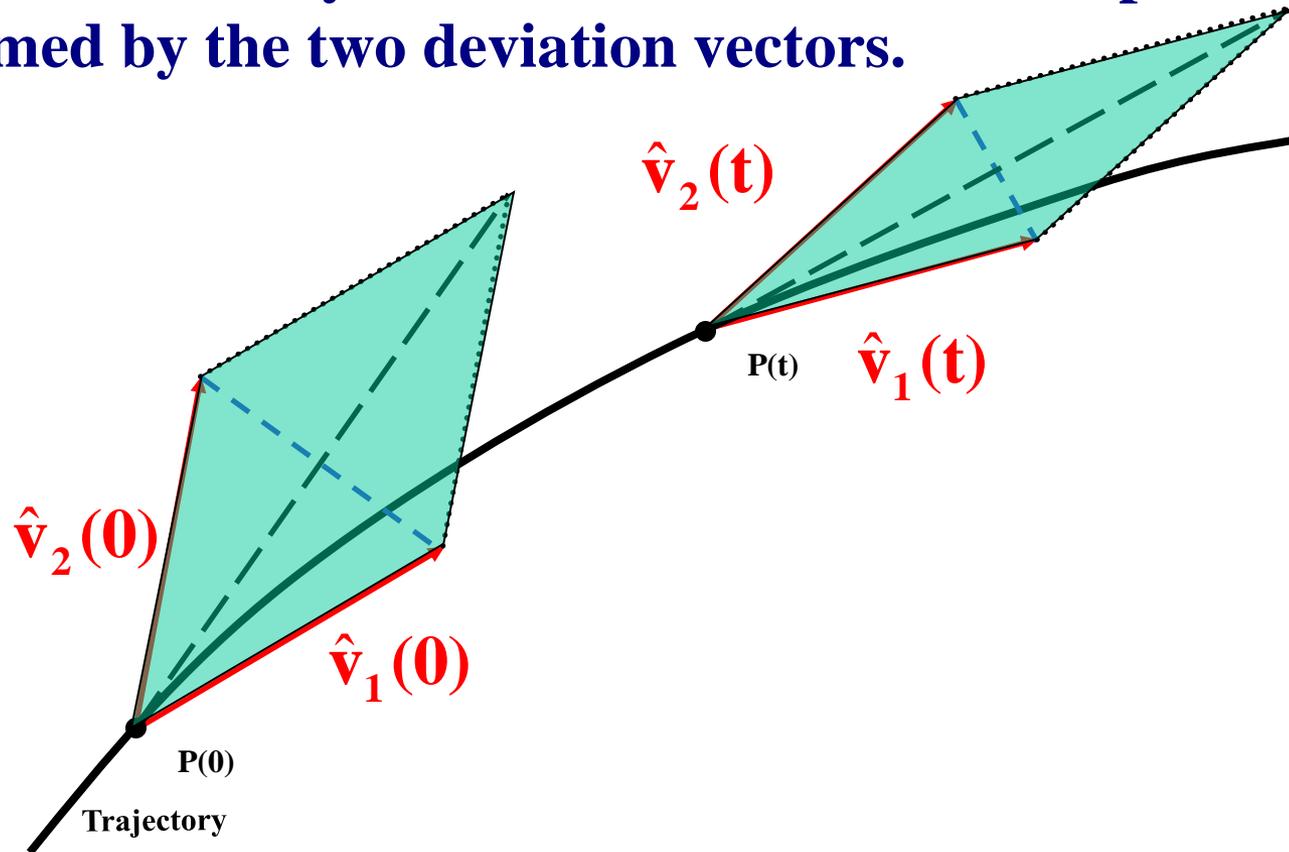
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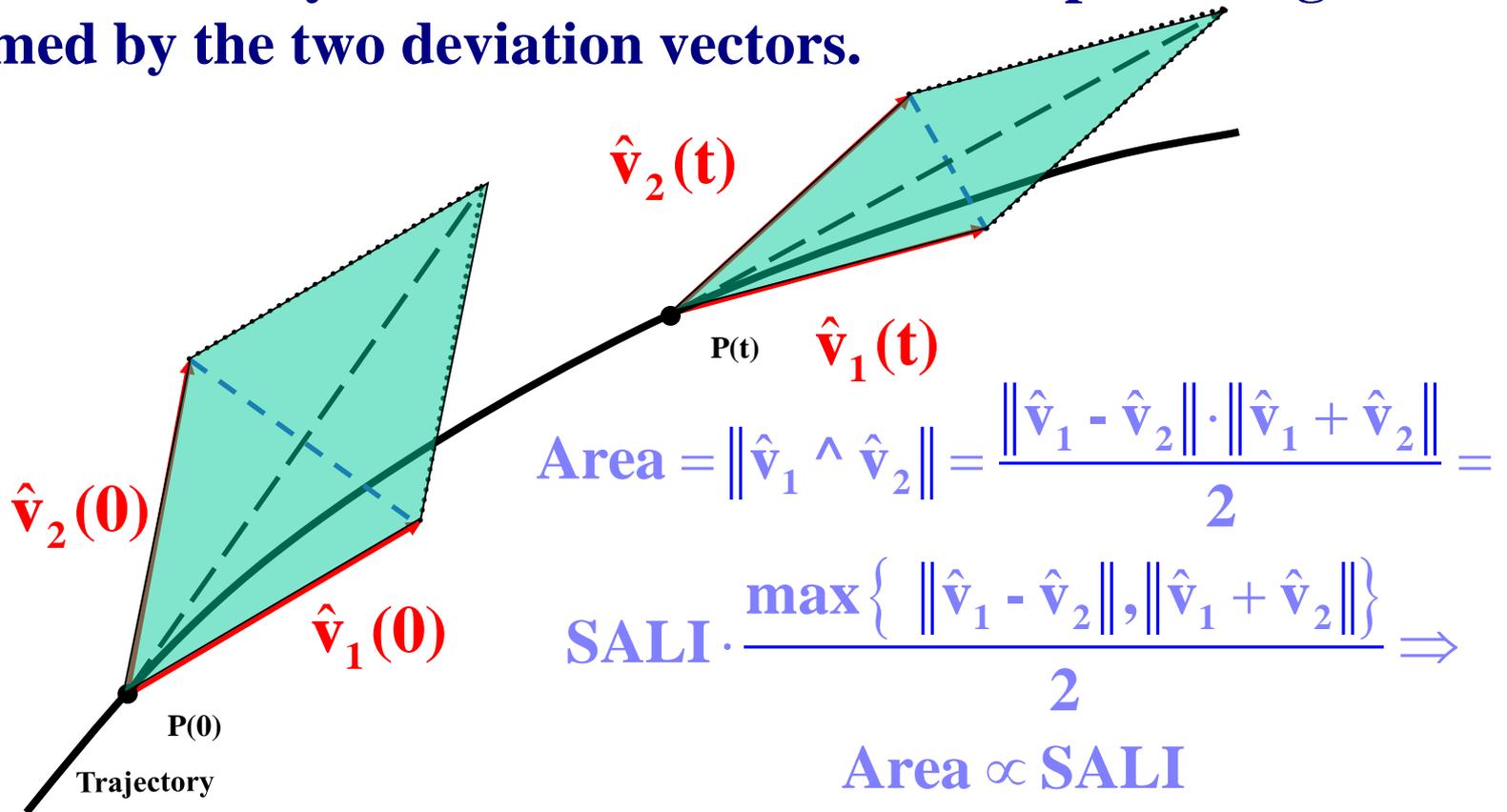
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Definition of Generalized Alignment Index (GALI)

SALI effectively measures the ‘area’ of the parallelogram formed by the two deviation vectors.



Definition of GALI

In the case of an N degree of freedom Hamiltonian system or a $2N$ symplectic map we follow the evolution of

k deviation vectors with $2 \leq k \leq 2N$,

and define (Ch.S., Bountis, Antonopoulos, 2007, Physica D) the Generalized Alignment Index (GALI) of order k :

$$\text{GALI}_k(\mathbf{t}) = \left\| \hat{\mathbf{v}}_1(\mathbf{t}) \wedge \hat{\mathbf{v}}_2(\mathbf{t}) \wedge \dots \wedge \hat{\mathbf{v}}_k(\mathbf{t}) \right\|$$

where

$$\hat{\mathbf{v}}_1(\mathbf{t}) = \frac{\mathbf{v}_1(\mathbf{t})}{\|\mathbf{v}_1(\mathbf{t})\|}$$

Wedge product

We consider as a basis of the $2N$ -dimensional tangent space of the system the usual set of orthonormal vectors:

$$\hat{\mathbf{e}}_1 = (1, 0, 0, \dots, 0), \hat{\mathbf{e}}_2 = (0, 1, 0, \dots, 0), \dots, \hat{\mathbf{e}}_{2N} = (0, 0, 0, \dots, 1)$$

Then for k deviation vectors we have:

$$\begin{bmatrix} \hat{\mathbf{v}}_1 \\ \hat{\mathbf{v}}_2 \\ \vdots \\ \hat{\mathbf{v}}_k \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{11} & \mathbf{v}_{12} & \cdots & \mathbf{v}_{12N} \\ \mathbf{v}_{21} & \mathbf{v}_{22} & \cdots & \mathbf{v}_{22N} \\ \vdots & \vdots & & \vdots \\ \mathbf{v}_{k1} & \mathbf{v}_{k2} & \cdots & \mathbf{v}_{k2N} \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \vdots \\ \hat{\mathbf{e}}_{2N} \end{bmatrix}$$

$$\hat{\mathbf{v}}_1 \wedge \hat{\mathbf{v}}_2 \wedge \cdots \wedge \hat{\mathbf{v}}_k = \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq 2N} \begin{vmatrix} \mathbf{v}_{1i_1} & \mathbf{v}_{1i_2} & \cdots & \mathbf{v}_{1i_k} \\ \mathbf{v}_{2i_1} & \mathbf{v}_{2i_2} & \cdots & \mathbf{v}_{2i_k} \\ \vdots & \vdots & & \vdots \\ \mathbf{v}_{ki_1} & \mathbf{v}_{ki_2} & \cdots & \mathbf{v}_{ki_k} \end{vmatrix} \hat{\mathbf{e}}_{i_1} \wedge \hat{\mathbf{e}}_{i_2} \wedge \cdots \wedge \hat{\mathbf{e}}_{i_k}$$

Norm of wedge product

We define as ‘norm’ of the wedge product the quantity :

$$\|\hat{\mathbf{v}}_1 \wedge \hat{\mathbf{v}}_2 \wedge \cdots \wedge \hat{\mathbf{v}}_k\| = \left\{ \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq 2N} \begin{vmatrix} \mathbf{v}_{1i_1} & \mathbf{v}_{1i_2} & \cdots & \mathbf{v}_{1i_k} \\ \mathbf{v}_{2i_1} & \mathbf{v}_{2i_2} & \cdots & \mathbf{v}_{2i_k} \\ \vdots & \vdots & & \vdots \\ \mathbf{v}_{ki_1} & \mathbf{v}_{ki_2} & \cdots & \mathbf{v}_{ki_k} \end{vmatrix}^2 \right\}^{1/2}$$

Computation of GALI - Example

Let us compute GALI_3 in the case of 2D Hamiltonian system (4-dimensional phase space).

$$\begin{bmatrix} \hat{\mathbf{V}}_1 \\ \hat{\mathbf{V}}_2 \\ \hat{\mathbf{V}}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{11} & \mathbf{v}_{12} & \mathbf{v}_{13} & \mathbf{v}_{14} \\ \mathbf{v}_{21} & \mathbf{v}_{22} & \mathbf{v}_{23} & \mathbf{v}_{24} \\ \mathbf{v}_{31} & \mathbf{v}_{32} & \mathbf{v}_{33} & \mathbf{v}_{34} \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \\ \hat{\mathbf{e}}_4 \end{bmatrix}$$

Computation of GALI - Example

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$$\text{GALI}_3 = \|\hat{\mathbf{v}}_1 \wedge \hat{\mathbf{v}}_2 \wedge \hat{\mathbf{v}}_3\| = \left\{ \begin{array}{ccc} \text{Columns } \mathbf{1} & \mathbf{2} & \mathbf{3} \\ \mathbf{v}_{11} & \mathbf{v}_{12} & \mathbf{v}_{13} \\ \mathbf{v}_{21} & \mathbf{v}_{22} & \mathbf{v}_{23} \\ \mathbf{v}_{31} & \mathbf{v}_{32} & \mathbf{v}_{33} \end{array} \right\}^2 +$$

Computation of GALI - Example

Let us compute GALI_3 in the case of 2D Hamiltonian system (4-dimensional phase space).

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$$\text{GALI}_3 = \|\hat{\mathbf{v}}_1 \wedge \hat{\mathbf{v}}_2 \wedge \hat{\mathbf{v}}_3\| = \left\{ \begin{array}{ccc} \text{Columns } 1 & 2 & 3 \\ \mathbf{v}_{11} & \mathbf{v}_{12} & \mathbf{v}_{13} \\ \mathbf{v}_{21} & \mathbf{v}_{22} & \mathbf{v}_{23} \\ \mathbf{v}_{31} & \mathbf{v}_{32} & \mathbf{v}_{33} \end{array} \right\}^2 + \left\{ \begin{array}{ccc} 1 & 2 & 4 \\ \mathbf{v}_{11} & \mathbf{v}_{12} & \mathbf{v}_{14} \\ \mathbf{v}_{21} & \mathbf{v}_{22} & \mathbf{v}_{24} \\ \mathbf{v}_{31} & \mathbf{v}_{32} & \mathbf{v}_{34} \end{array} \right\}^2 +$$

Computation of GALI - Example

Let us compute GALI_3 in the case of 2D Hamiltonian system (4-dimensional phase space).

$$\begin{bmatrix} \hat{\mathbf{v}}_1 \\ \hat{\mathbf{v}}_2 \\ \hat{\mathbf{v}}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{11} & \mathbf{v}_{12} & \mathbf{v}_{13} & \mathbf{v}_{14} \\ \mathbf{v}_{21} & \mathbf{v}_{22} & \mathbf{v}_{23} & \mathbf{v}_{24} \\ \mathbf{v}_{31} & \mathbf{v}_{32} & \mathbf{v}_{33} & \mathbf{v}_{34} \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \\ \hat{\mathbf{e}}_4 \end{bmatrix}$$

$$\text{GALI}_3 = \|\hat{\mathbf{v}}_1 \wedge \hat{\mathbf{v}}_2 \wedge \hat{\mathbf{v}}_3\| = \left\{ \begin{array}{c} \text{Columns } \mathbf{1} \quad \mathbf{2} \quad \mathbf{3} \\ \left| \begin{array}{ccc} \mathbf{v}_{11} & \mathbf{v}_{12} & \mathbf{v}_{13} \\ \mathbf{v}_{21} & \mathbf{v}_{22} & \mathbf{v}_{23} \\ \mathbf{v}_{31} & \mathbf{v}_{32} & \mathbf{v}_{33} \end{array} \right|^2 + \left| \begin{array}{ccc} \mathbf{1} \quad \mathbf{2} \quad \mathbf{4} \\ \mathbf{v}_{11} & \mathbf{v}_{12} & \mathbf{v}_{14} \\ \mathbf{v}_{21} & \mathbf{v}_{22} & \mathbf{v}_{24} \\ \mathbf{v}_{31} & \mathbf{v}_{32} & \mathbf{v}_{34} \end{array} \right|^2 + \\ \left| \begin{array}{ccc} \mathbf{1} \quad \mathbf{3} \quad \mathbf{4} \\ \mathbf{v}_{11} & \mathbf{v}_{13} & \mathbf{v}_{14} \\ \mathbf{v}_{21} & \mathbf{v}_{23} & \mathbf{v}_{24} \\ \mathbf{v}_{31} & \mathbf{v}_{33} & \mathbf{v}_{34} \end{array} \right|^2 + \end{array} \right.$$

Computation of GALI - Example

Let us compute GALI_3 in the case of 2D Hamiltonian system (4-dimensional phase space).

$$\begin{bmatrix} \hat{\mathbf{v}}_1 \\ \hat{\mathbf{v}}_2 \\ \hat{\mathbf{v}}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{11} & \mathbf{v}_{12} & \mathbf{v}_{13} & \mathbf{v}_{14} \\ \mathbf{v}_{21} & \mathbf{v}_{22} & \mathbf{v}_{23} & \mathbf{v}_{24} \\ \mathbf{v}_{31} & \mathbf{v}_{32} & \mathbf{v}_{33} & \mathbf{v}_{34} \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \hat{\mathbf{e}}_3 \\ \hat{\mathbf{e}}_4 \end{bmatrix}$$

$$\text{GALI}_3 = \|\hat{\mathbf{v}}_1 \wedge \hat{\mathbf{v}}_2 \wedge \hat{\mathbf{v}}_3\| = \left\{ \begin{array}{l} \text{Columns } \mathbf{1} \quad \mathbf{2} \quad \mathbf{3} \\ \left| \begin{array}{ccc} \mathbf{v}_{11} & \mathbf{v}_{12} & \mathbf{v}_{13} \\ \mathbf{v}_{21} & \mathbf{v}_{22} & \mathbf{v}_{23} \\ \mathbf{v}_{31} & \mathbf{v}_{32} & \mathbf{v}_{33} \end{array} \right|^2 + \left| \begin{array}{ccc} \mathbf{1} \quad \mathbf{2} \quad \mathbf{4} \\ \mathbf{v}_{11} & \mathbf{v}_{12} & \mathbf{v}_{14} \\ \mathbf{v}_{21} & \mathbf{v}_{22} & \mathbf{v}_{24} \\ \mathbf{v}_{31} & \mathbf{v}_{32} & \mathbf{v}_{34} \end{array} \right|^2 + \\ \left. \left| \begin{array}{ccc} \mathbf{1} \quad \mathbf{3} \quad \mathbf{4} \\ \mathbf{v}_{11} & \mathbf{v}_{13} & \mathbf{v}_{14} \\ \mathbf{v}_{21} & \mathbf{v}_{23} & \mathbf{v}_{24} \\ \mathbf{v}_{31} & \mathbf{v}_{33} & \mathbf{v}_{34} \end{array} \right|^2 + \left| \begin{array}{ccc} \mathbf{2} \quad \mathbf{3} \quad \mathbf{4} \\ \mathbf{v}_{12} & \mathbf{v}_{13} & \mathbf{v}_{14} \\ \mathbf{v}_{22} & \mathbf{v}_{23} & \mathbf{v}_{24} \\ \mathbf{v}_{32} & \mathbf{v}_{33} & \mathbf{v}_{34} \end{array} \right|^2 \right\}^{1/2}$$

Efficient computation of GALI

For k deviation vectors:

$$\begin{bmatrix} \hat{\mathbf{v}}_1 \\ \hat{\mathbf{v}}_2 \\ \vdots \\ \hat{\mathbf{v}}_k \end{bmatrix} = \begin{bmatrix} \mathbf{v}_{11} & \mathbf{v}_{12} & \cdots & \mathbf{v}_{12N} \\ \mathbf{v}_{21} & \mathbf{v}_{22} & \cdots & \mathbf{v}_{22N} \\ \vdots & \vdots & & \vdots \\ \mathbf{v}_{k1} & \mathbf{v}_{k2} & \cdots & \mathbf{v}_{k2N} \end{bmatrix} \cdot \begin{bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \vdots \\ \hat{\mathbf{e}}_{2N} \end{bmatrix} = \mathbf{A} \cdot \begin{bmatrix} \hat{\mathbf{e}}_1 \\ \hat{\mathbf{e}}_2 \\ \vdots \\ \hat{\mathbf{e}}_{2N} \end{bmatrix}$$

the 'norm' of the wedge product is given by:

$$\|\hat{\mathbf{v}}_1 \wedge \hat{\mathbf{v}}_2 \wedge \cdots \wedge \hat{\mathbf{v}}_k\| = \left\{ \sum_{1 \leq i_1 < i_2 < \cdots < i_k \leq 2N} \begin{vmatrix} \mathbf{v}_{1i_1} & \mathbf{v}_{1i_2} & \cdots & \mathbf{v}_{1i_k} \\ \mathbf{v}_{2i_1} & \mathbf{v}_{2i_2} & \cdots & \mathbf{v}_{2i_k} \\ \vdots & \vdots & & \vdots \\ \mathbf{v}_{ki_1} & \mathbf{v}_{ki_2} & \cdots & \mathbf{v}_{ki_k} \end{vmatrix}^2 \right\}^{1/2} = \sqrt{\det(\mathbf{A} \cdot \mathbf{A}^T)}$$

Efficient computation of GALI

From **Singular Value Decomposition (SVD)** of A^T we get:

$$A^T = U \cdot W \cdot V^T$$

where U is a column-orthogonal $2N \times k$ matrix ($U^T \cdot U = I$), V^T is a $k \times k$ orthogonal matrix ($V \cdot V^T = I$), and W is a diagonal $k \times k$ matrix with positive or zero elements, the so-called **singular values**. So, we get:

$$\det(A \cdot A^T) = \det(V \cdot W^T \cdot U^T \cdot U \cdot W \cdot V^T) = \det(V \cdot W \cdot I \cdot W \cdot V^T) =$$

$$\det(V \cdot W^2 \cdot V^T) = \det(V \cdot \text{diag}(w_1^2, w_2^2, \dots, w_k^2) \cdot V^T) = \prod_{i=1}^k w_i^2$$

Thus, $GALI_k$ is computed by:

$$GALI_k = \sqrt{\det(A \cdot A^T)} = \prod_{i=1}^k w_i \Rightarrow \log(GALI_k) = \sum_{i=1}^k \log(w_i)$$

Behavior of $GALI_k$ for chaotic motion

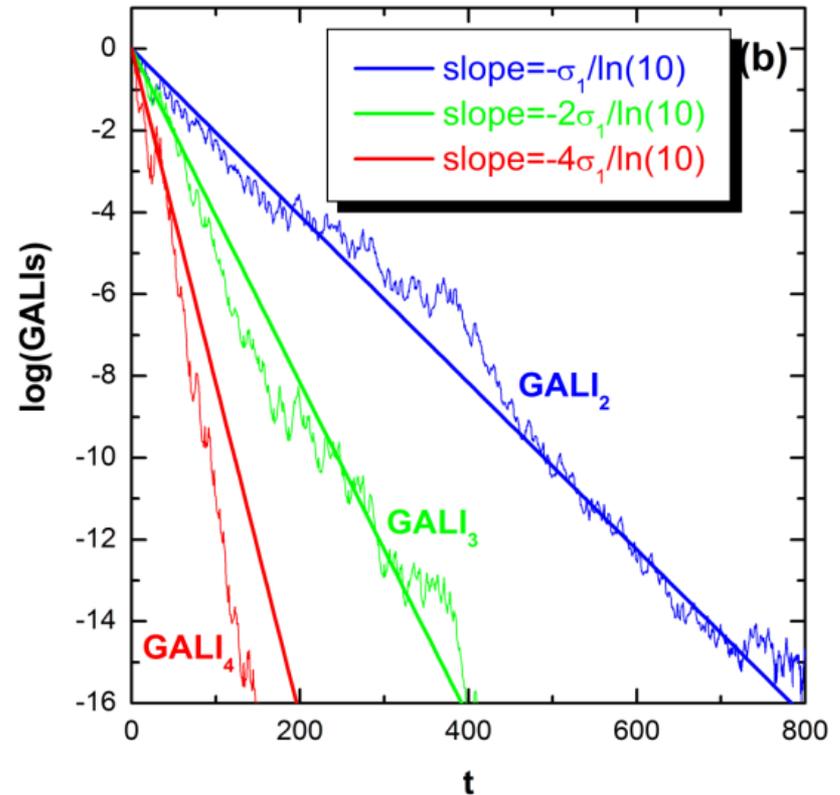
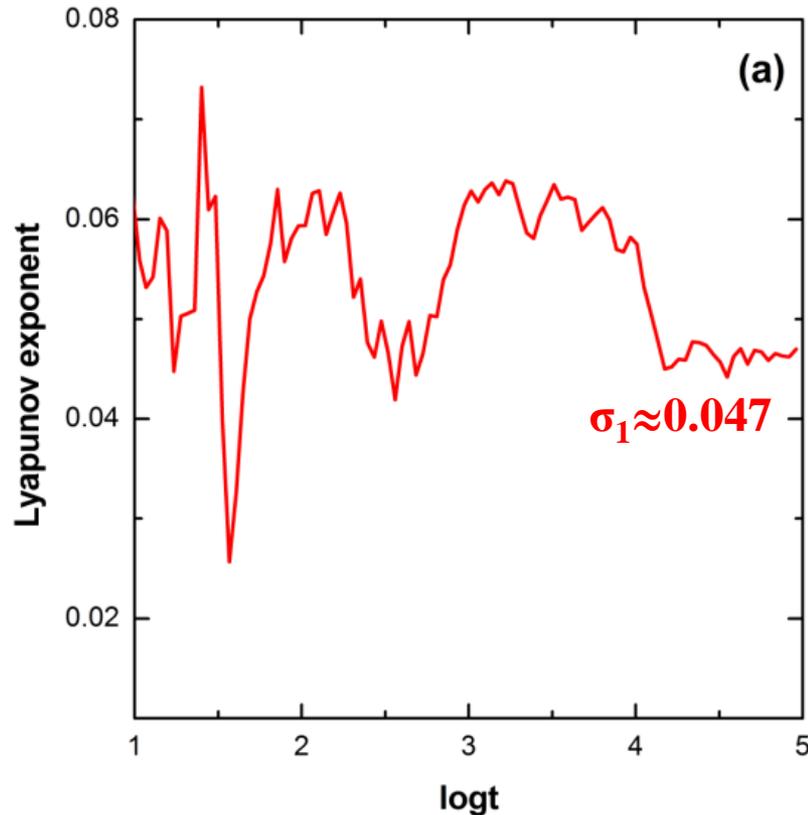
$GALI_k$ ($2 \leq k \leq 2N$) tends exponentially to zero with exponents that involve the values of the first k largest Lyapunov exponents $\sigma_1, \sigma_2, \dots, \sigma_k$:

$$GALI_k(t) \propto e^{-[(\sigma_1 - \sigma_2) + (\sigma_1 - \sigma_3) + \dots + (\sigma_1 - \sigma_k)]t}$$

The above relation is valid even if some Lyapunov exponents are equal, or very close to each other.

Behavior of $GALI_k$ for chaotic motion

2D Hamiltonian (Hénon-Heiles system)

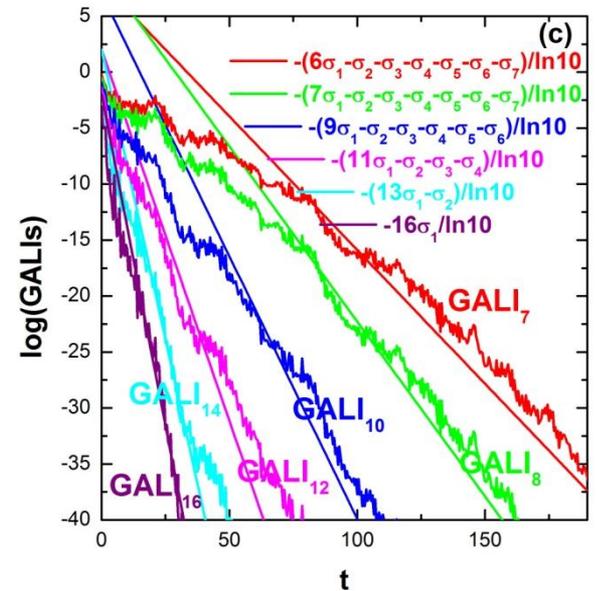
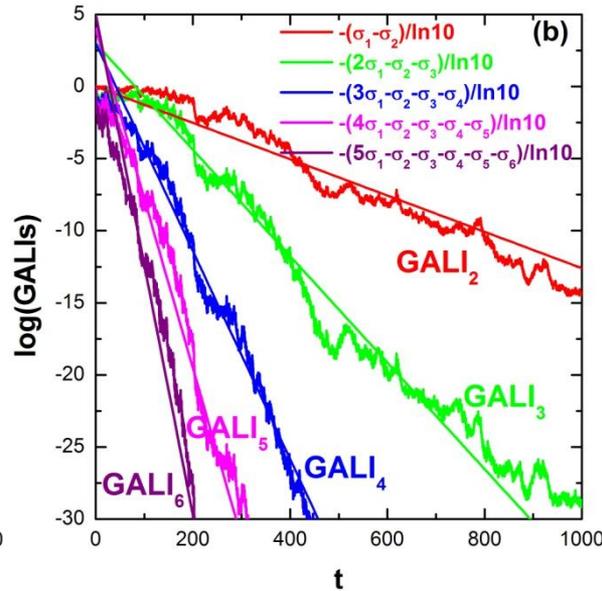
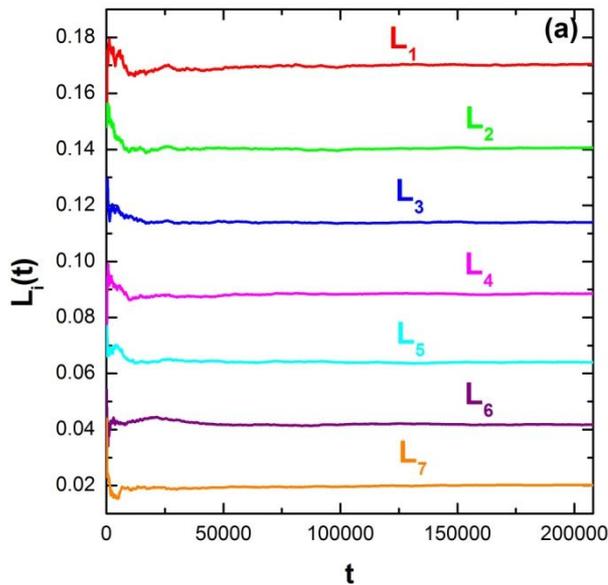


Behavior of $GALI_k$ for chaotic motion

N particles Fermi-Pasta-Ulam (FPU) system:

$$H = \frac{1}{2} \sum_{i=1}^N p_i^2 + \sum_{i=0}^N \left[\frac{1}{2} (q_{i+1} - q_i)^2 + \frac{\beta}{4} (q_{i+1} - q_i)^4 \right]$$

with fixed boundary conditions, $N=8$ and $\beta=1.5$.



Behavior of $GALI_k$ for regular motion

If the motion occurs on an s -dimensional torus with $s \leq N$ then the behavior of $GALI_k$ is given by (Ch.S., Bountis, Antonopoulos, 2008, Eur. Phys. J. Sp. Top.):

$$GALI_k(t) \propto \begin{cases} \text{constant} & \text{if } 2 \leq k \leq s \\ \frac{1}{t^{k-s}} & \text{if } s < k \leq 2N - s \\ \frac{1}{t^{2(k-N)}} & \text{if } 2N - s < k \leq 2N \end{cases}$$

while in the common case with $s=N$ we have :

$$GALI_k(t) \propto \begin{cases} \text{constant} & \text{if } 2 \leq k \leq N \\ \frac{1}{t^{2(k-N)}} & \text{if } N < k \leq 2N \end{cases}$$

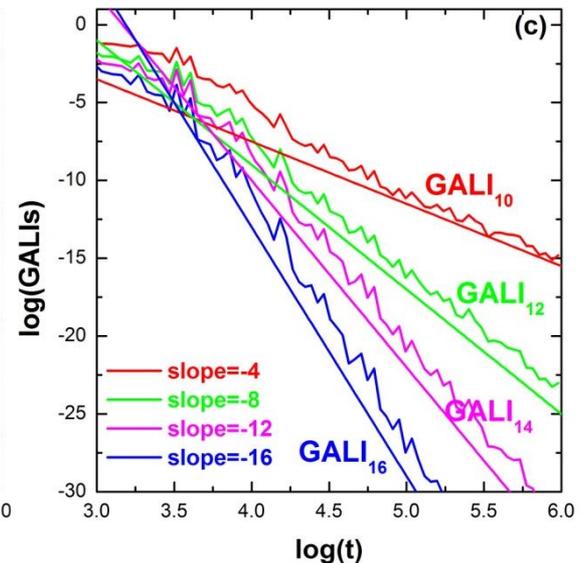
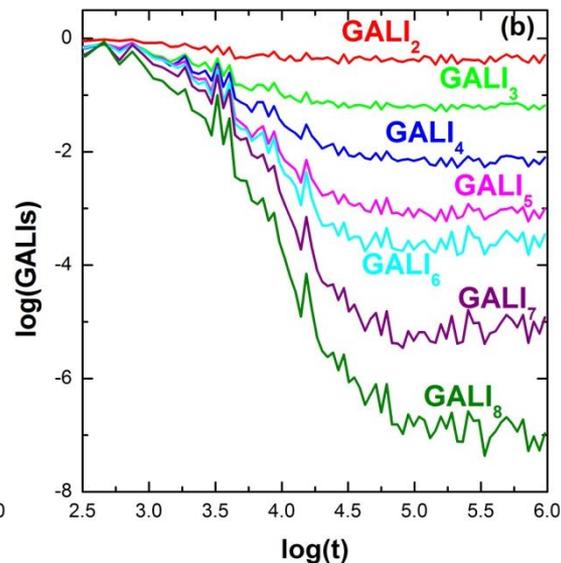
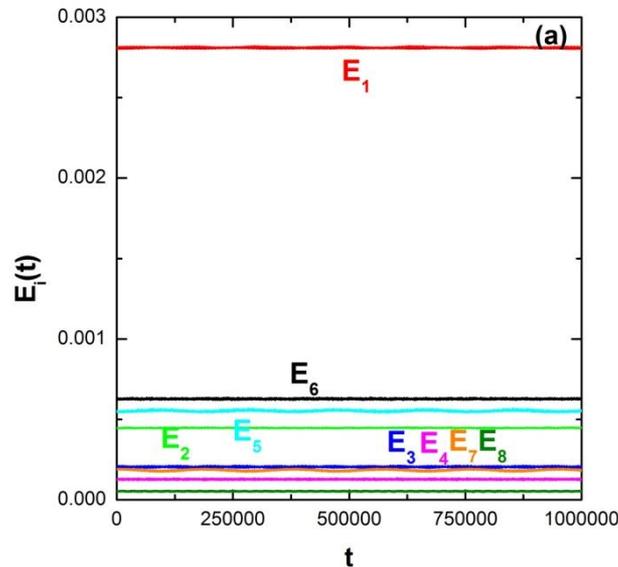
Behavior of $GALI_k$ for regular motion

N=8 FPU system: The unperturbed Hamiltonian ($\beta=0$) is written as a sum of the so-called **harmonic energies E_i** :

$$E_i = \frac{1}{2} (P_i^2 + \omega_i^2 Q_i^2), \quad i = 1, \dots, N$$

with:

$$Q_i = \sqrt{\frac{2}{N+1}} \sum_{k=1}^N q_k \sin\left(\frac{ki\pi}{N+1}\right), \quad P_i = \sqrt{\frac{2}{N+1}} \sum_{k=1}^N p_k \sin\left(\frac{ki\pi}{N+1}\right), \quad \omega_i = 2 \sin\left(\frac{i\pi}{2(N+1)}\right)$$



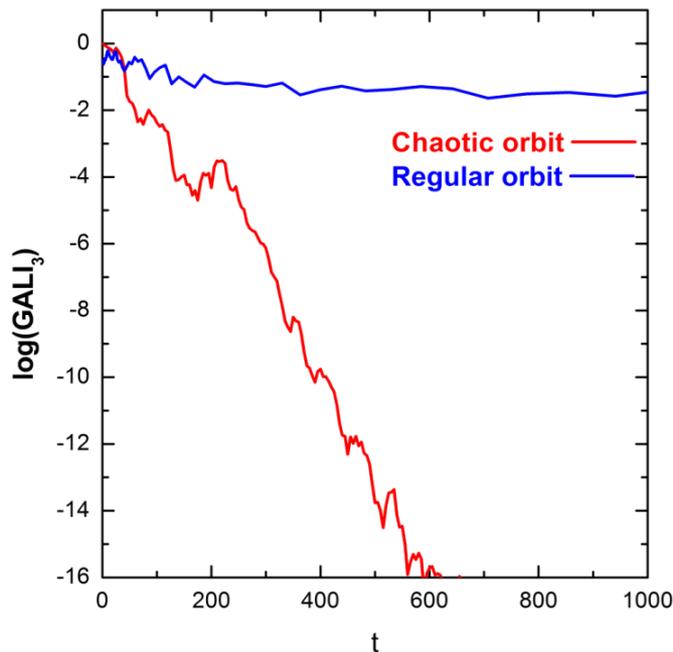
Global dynamics

- $GALI_2$ (practically equivalent to the use of SALI)

- $GALI_N$

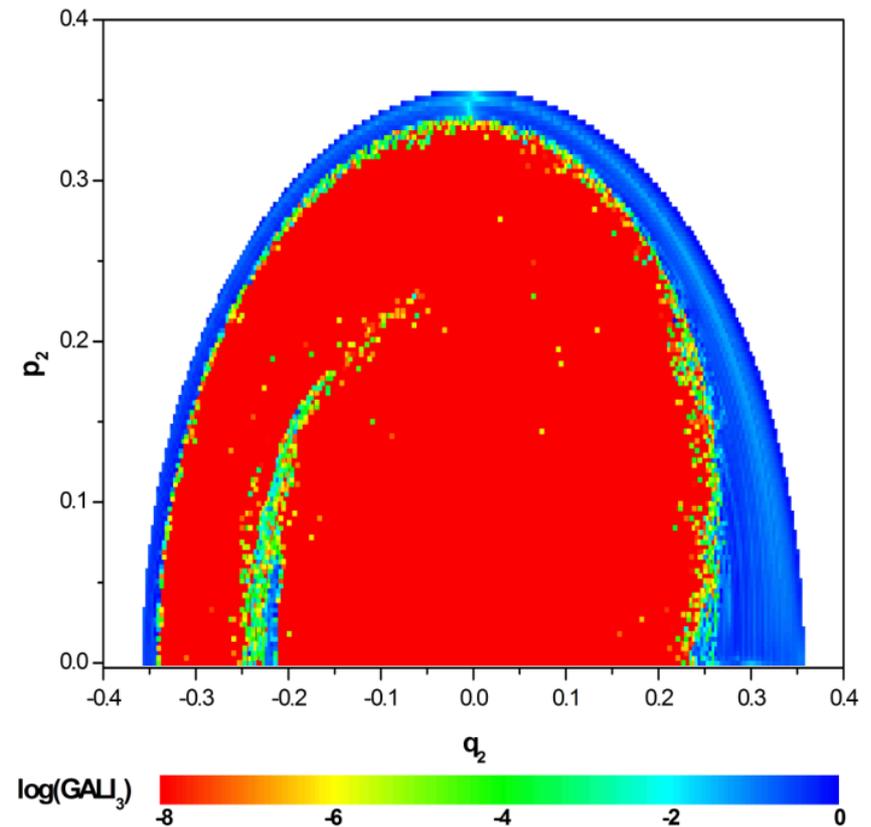
**Chaotic motion: $GALI_N \rightarrow 0$
(exponential decay)**

**Regular motion:
 $GALI_N \rightarrow \text{constant} \neq 0$**



3D Hamiltonian

Subspace $q_3=p_3=0$, $p_2 \geq 0$ for $t=1000$.



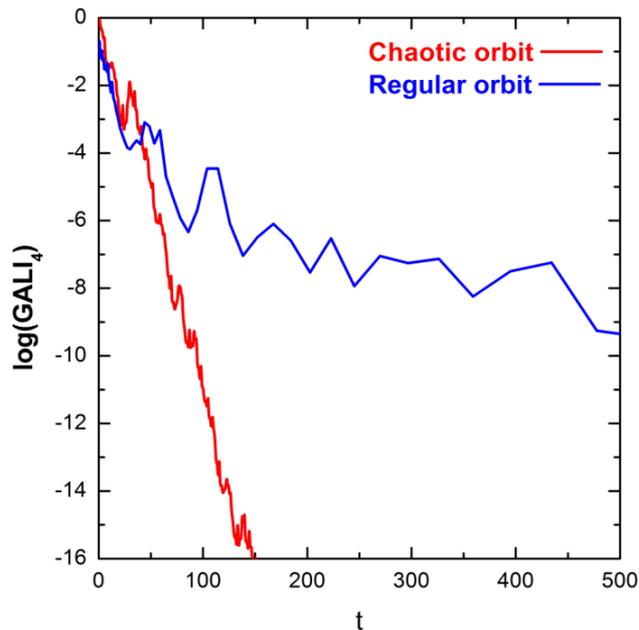
Global dynamics

$GALI_k$ with $k > N$

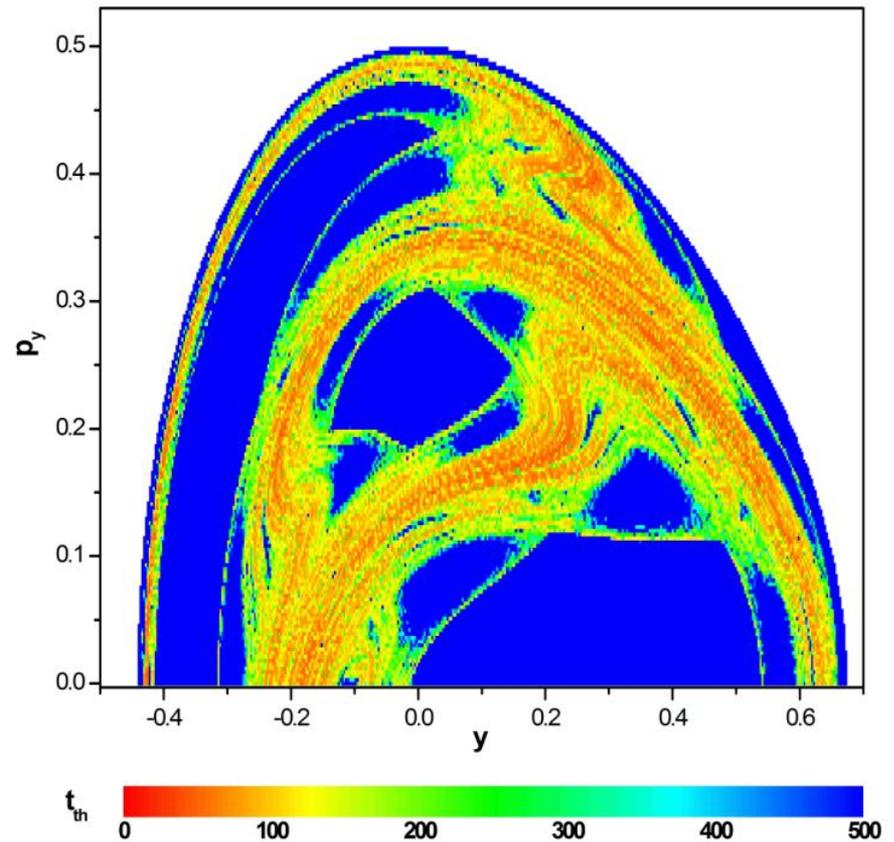
The index tends to zero both for regular and chaotic orbits but with completely different time rates:

Chaotic motion: exponential decay

Regular motion: power law



2D Hamiltonian (Hénon-Heiles)
Time needed for $GALI_4 < 10^{-12}$



Behavior of $GALI_k$

Chaotic motion:

$GALI_k \rightarrow 0$ exponential decay

$$GALI_k(t) \propto e^{-[(\sigma_1 - \sigma_2) + (\sigma_1 - \sigma_3) + \dots + (\sigma_1 - \sigma_k)]t}$$

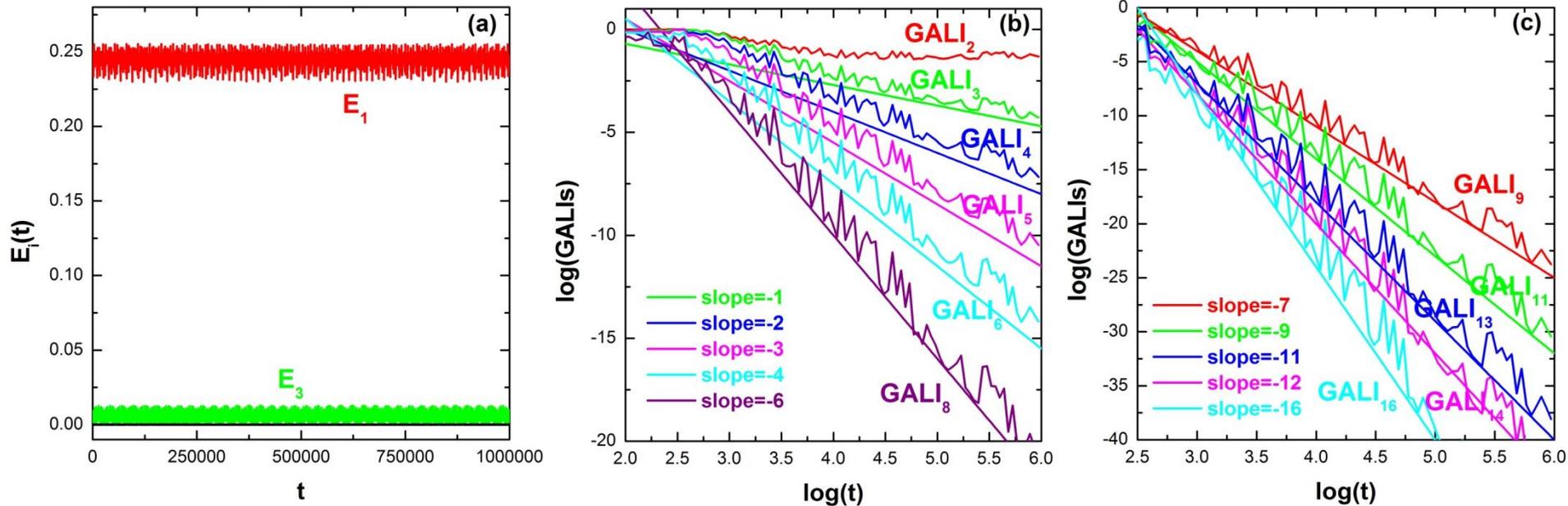
Regular motion:

$GALI_k \rightarrow \text{constant} \neq 0$ or $GALI_k \rightarrow 0$ power law decay

$$GALI_k(t) \propto \begin{cases} \text{constant} & \text{if } 2 \leq k \leq s \\ \frac{1}{t^{k-s}} & \text{if } s < k \leq 2N - s \\ \frac{1}{t^{2(k-N)}} & \text{if } 2N - s < k \leq 2N \end{cases}$$

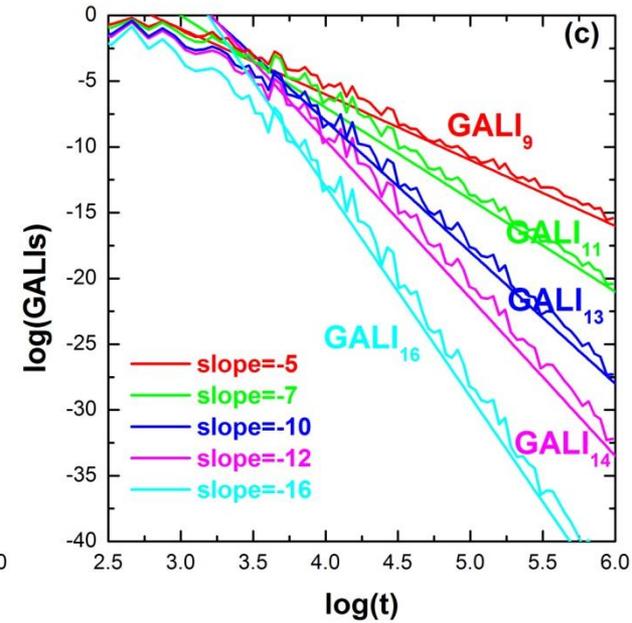
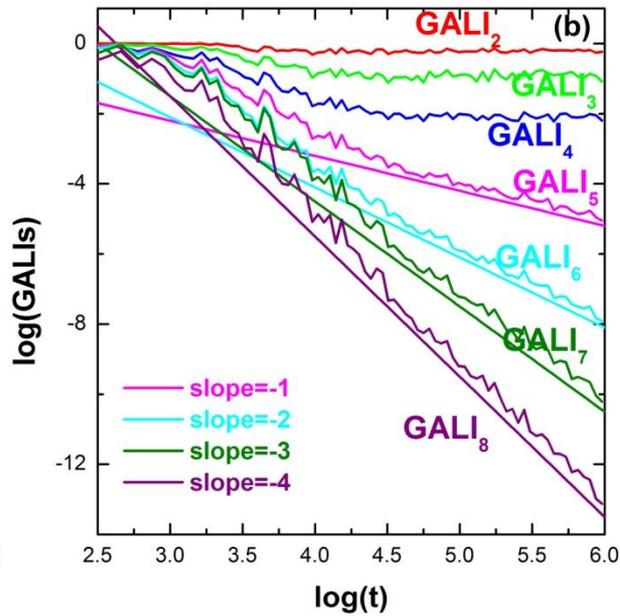
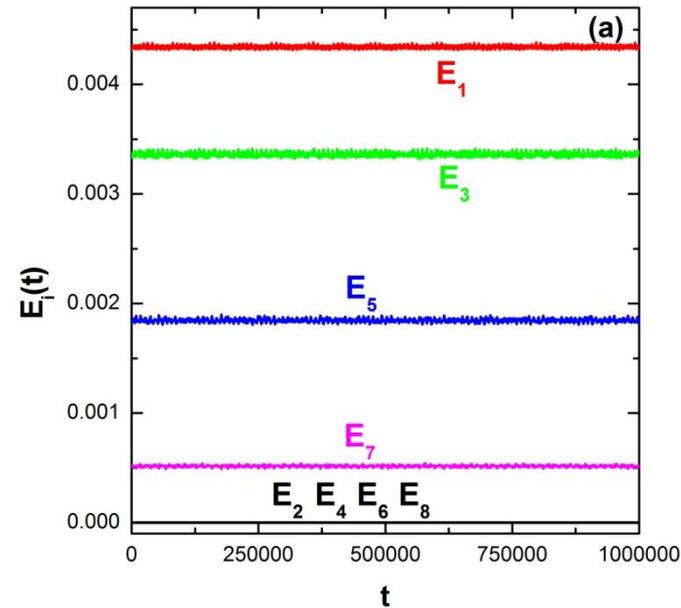
Regular motion on low-dimensional tori

A regular orbit lying on a **2-dimensional torus** for the $N=8$ FPU system.



Regular motion on low-dimensional tori

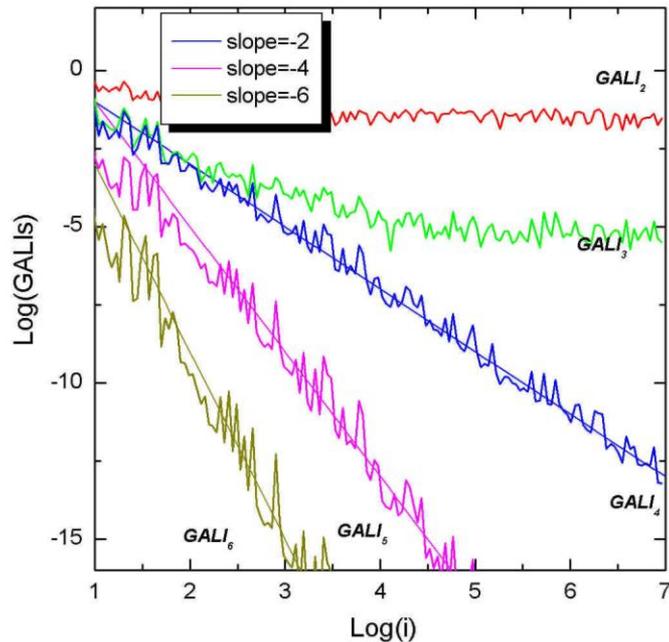
A regular orbit lying on a **4-dimensional torus** for the $N=8$ FPU system.



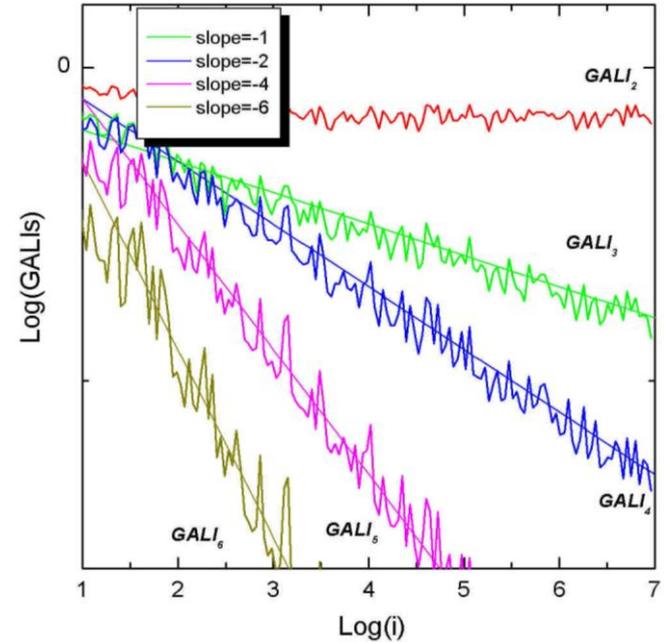
Low-dimensional tori - 6D map

$$\begin{aligned}
 \mathbf{x}'_1 &= \mathbf{x}_1 + \mathbf{x}'_2 \\
 \mathbf{x}'_2 &= \mathbf{x}_2 + \frac{K_1}{2\pi} \sin(2\pi\mathbf{x}_1) - \frac{B}{2\pi} \{ \sin[2\pi(\mathbf{x}_5 - \mathbf{x}_1)] + \sin[2\pi(\mathbf{x}_3 - \mathbf{x}_1)] \} \\
 \mathbf{x}'_3 &= \mathbf{x}_3 + \mathbf{x}'_4 \\
 \mathbf{x}'_4 &= \mathbf{x}_4 + \frac{K_2}{2\pi} \sin(2\pi\mathbf{x}_3) - \frac{B}{2\pi} \{ \sin[2\pi(\mathbf{x}_1 - \mathbf{x}_3)] + \sin[2\pi(\mathbf{x}_5 - \mathbf{x}_3)] \} \pmod{1} \\
 \mathbf{x}'_5 &= \mathbf{x}_5 + \mathbf{x}'_6 \\
 \mathbf{x}'_6 &= \mathbf{x}_6 + \frac{K_3}{2\pi} \sin(2\pi\mathbf{x}_5) - \frac{B}{2\pi} \{ \sin[2\pi(\mathbf{x}_3 - \mathbf{x}_5)] + \sin[2\pi(\mathbf{x}_1 - \mathbf{x}_5)] \}
 \end{aligned}$$

3D torus

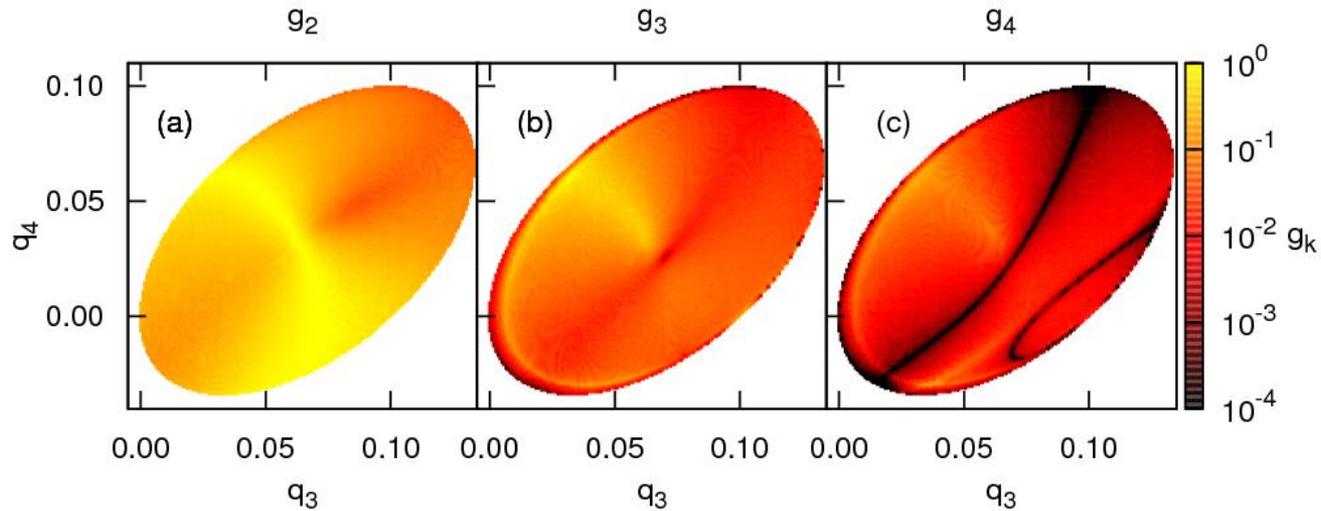


2D torus



Locating low-dimensional tori

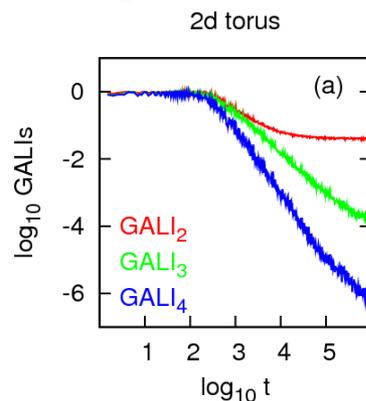
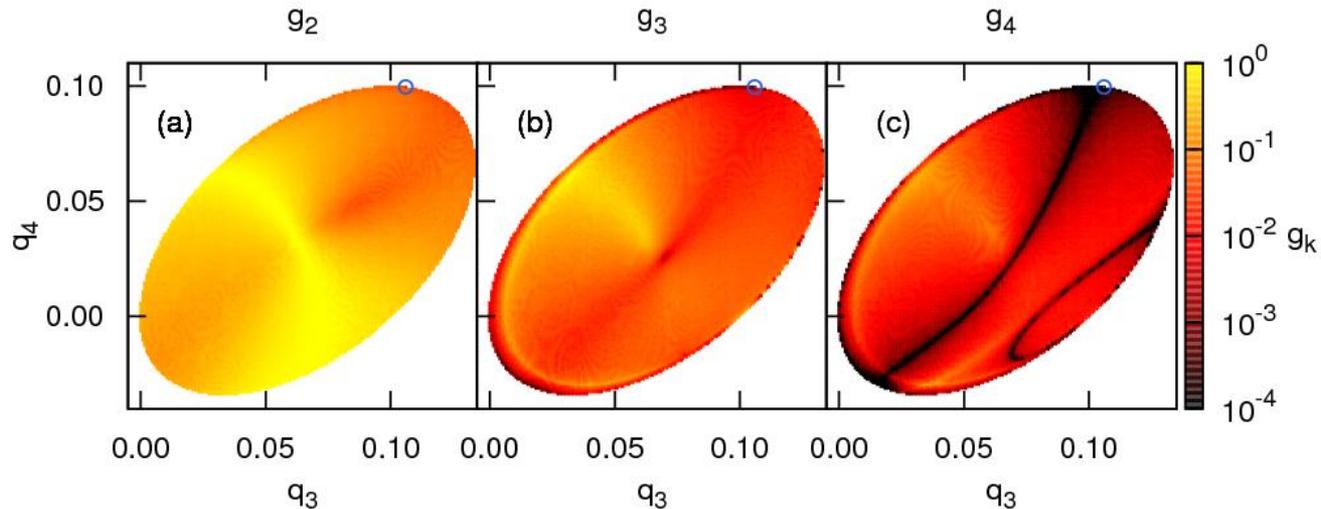
Orbits with $q_1=q_2=0.1$, $p_1=p_2=p_3=0$, $H=0.010075$ for the $N=4$ FPU system (Gerlach, Eggl, Ch.S., 2012, Int. J. Bifur. Chaos).



$$g_k = \frac{\text{GALI}_k}{\max(\text{GALI}_k)}$$

Locating low-dimensional tori

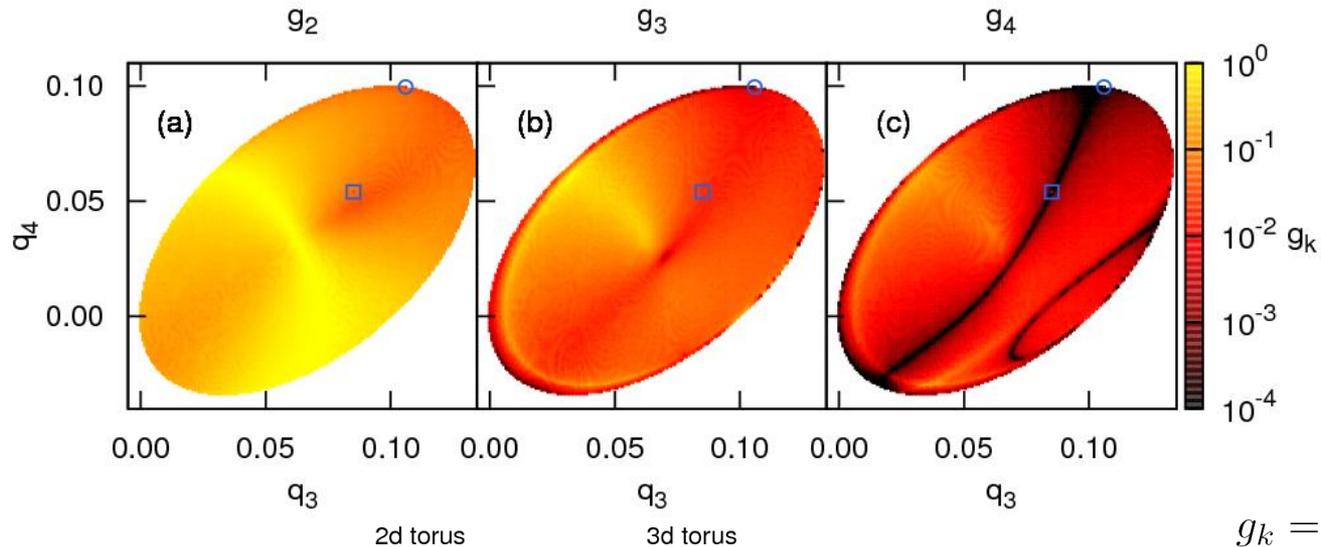
Orbits with $q_1=q_2=0.1$, $p_1=p_2=p_3=0$, $H=0.010075$ for the $N=4$ FPU system (Gerlach, Eggl, Ch.S., 2012, Int. J. Bifur. Chaos).



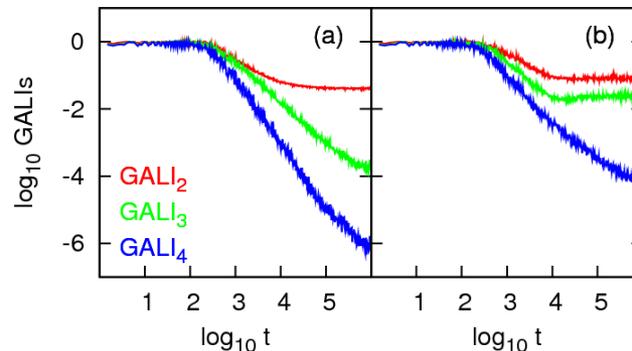
$$g_k = \frac{\text{GALI}_k}{\max(\text{GALI}_k)}$$

Locating low-dimensional tori

Orbits with $q_1=q_2=0.1$, $p_1=p_2=p_3=0$, $H=0.010075$ for the $N=4$ FPU system (Gerlach, Eggl, Ch.S., 2012, Int. J. Bifur. Chaos).

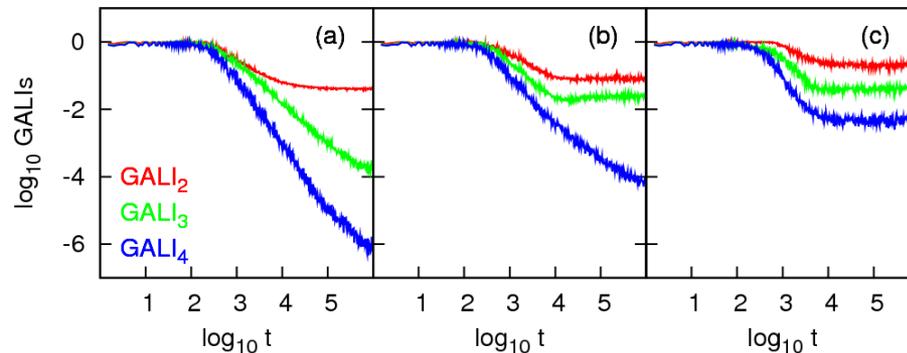
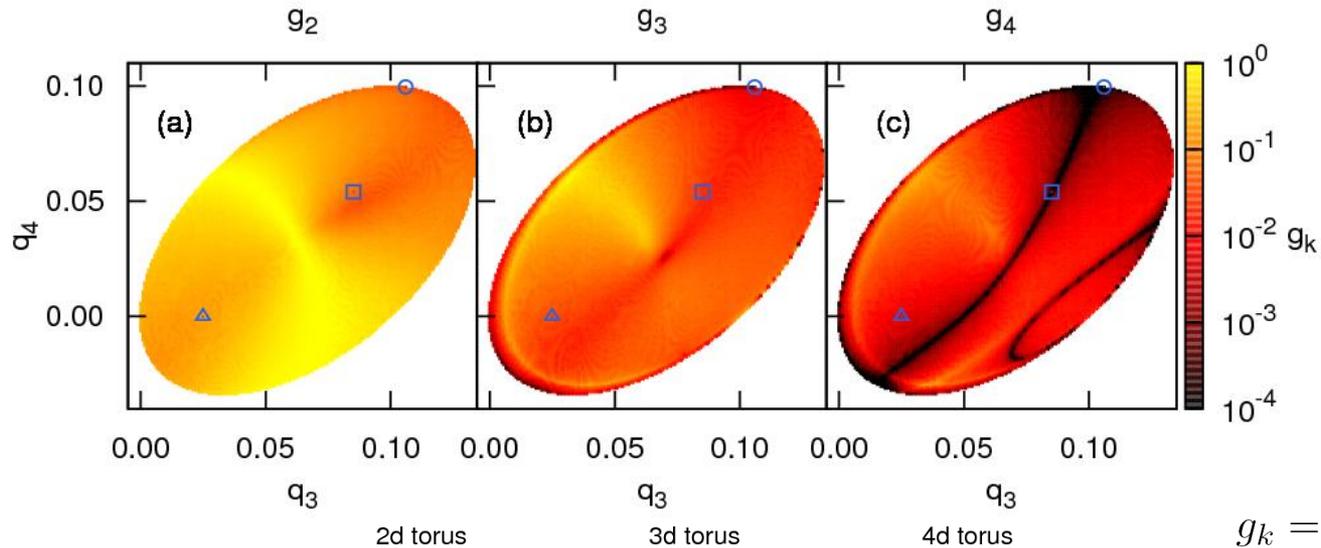


$$g_k = \frac{\text{GALI}_k}{\max(\text{GALI}_k)}$$



Locating low-dimensional tori

Orbits with $q_1=q_2=0.1$, $p_1=p_2=p_3=0$, $H=0.010075$ for the $N=4$ FPU system (Gerlach, Eggl, Ch.S., 2012, Int. J. Bifur. Chaos).



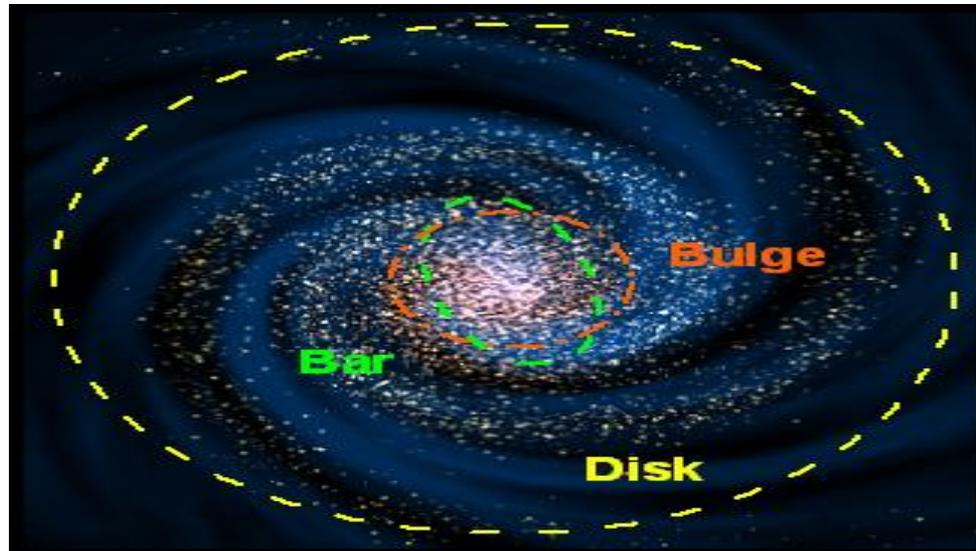
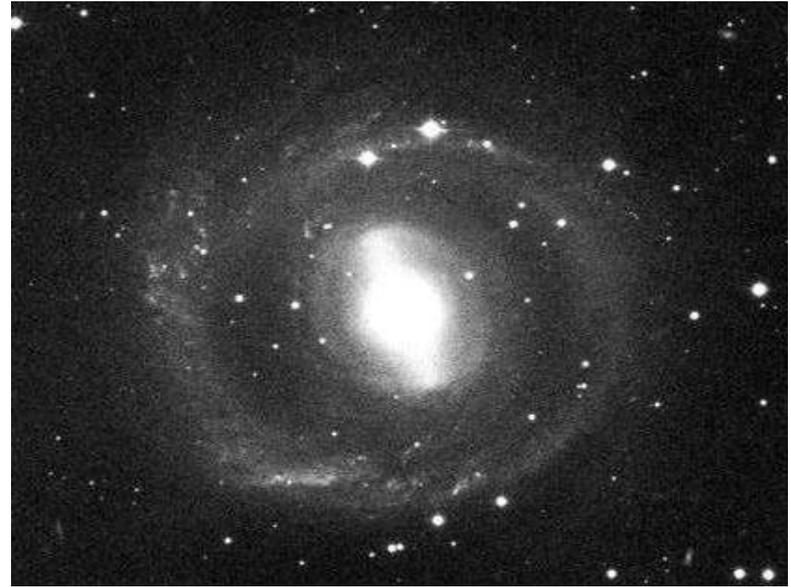
$$g_k = \frac{\text{GALI}_k}{\max(\text{GALI}_k)}$$

Barred galaxies

NGC 1433



NGC 2217



Barred galaxy model

The 3D bar rotates around its short z -axis (x : long axis and y : intermediate). The Hamiltonian that describes the motion for this model is:

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z) - \Omega_b(xp_y - yp_x) \equiv \text{Energy}$$

This model consists of the superposition of potentials describing an **axisymmetric** part and a **bar** component of the galaxy (**Manos, Bountis, Ch.S., 2013, J. Phys. A**).

a) Axisymmetric component:

i) Plummer sphere:

$$V_{\text{sphere}}(x, y, z) = -\frac{GM_S}{\sqrt{x^2 + y^2 + z^2 + \epsilon_s^2}}$$

ii) Miyamoto–Nagai disc:

$$V_{\text{disc}}(x, y, z) = -\frac{GM_D}{\sqrt{x^2 + y^2 + (A + \sqrt{B^2 + z^2})^2}}$$

b) Bar component: $V_{\text{bar}}(x, y, z) = -\pi Gabc \frac{\rho_c}{n+1} \int_{\lambda}^{\infty} \frac{du}{\Delta(u)} (1 - m^2(u))^{n+1}$,

(Ferrers bar)

$$\rho_c = \frac{105}{32\pi} \frac{GM_B}{abc}$$

where $m^2(u) = \frac{x^2}{a^2 + u} + \frac{y^2}{b^2 + u} + \frac{z^2}{c^2 + u}$, $\Delta^2(u) = (a^2 + u)(b^2 + u)(c^2 + u)$,

n : positive integer ($n = 2$ for our model), λ : the unique positive solution of $m^2(\lambda) = 1$

Its density is:

$$\rho = \begin{cases} \rho_c (1 - m^2)^n, & \text{for } m \leq 1 \\ 0, & \text{for } m > 1 \end{cases}, \text{ where } m^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}, \text{ } a > b > c \text{ and } n = 2.$$

Time-dependent barred galaxy model

The 3D bar rotates around its short z -axis (x : long axis and y : intermediate). The Hamiltonian that describes the motion for this model is:

$$H = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2) + V(x, y, z, t) - \Omega_b(xp_y - yp_x) \equiv \text{Energy}$$

This model consists of the superposition of potentials describing an **axisymmetric** part and a **bar** component of the galaxy (**Manos, Bountis, Ch.S., 2013, J. Phys. A**).

a) Axisymmetric component:

$$M_S + M_B(t) + M_D(t) = 1, \text{ with } M_B(t) = M_B(0) + \alpha t$$

i) Plummer sphere:

$$V_{sphere}(x, y, z) = -\frac{GM_S}{\sqrt{x^2 + y^2 + z^2 + \epsilon_s^2}}$$

ii) Miyamoto–Nagai disc:

$$V_{disc}(x, y, z) = -\frac{GM_D(t)}{\sqrt{x^2 + y^2 + (A + \sqrt{B^2 + z^2})^2}}$$

b) Bar component: $V_{bar}(x, y, z) = -\pi Gabc \frac{\rho_c}{n+1} \int_{\lambda}^{\infty} \frac{du}{\Delta(u)} (1 - m^2(u))^{n+1}$,

(Ferrers bar)

$$\text{where } m^2(u) = \frac{x^2}{a^2 + u} + \frac{y^2}{b^2 + u} + \frac{z^2}{c^2 + u}, \Delta^2(u) = (a^2 + u)(b^2 + u)(c^2 + u),$$

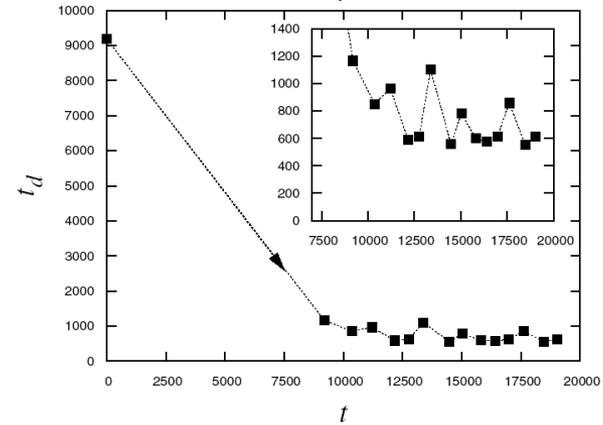
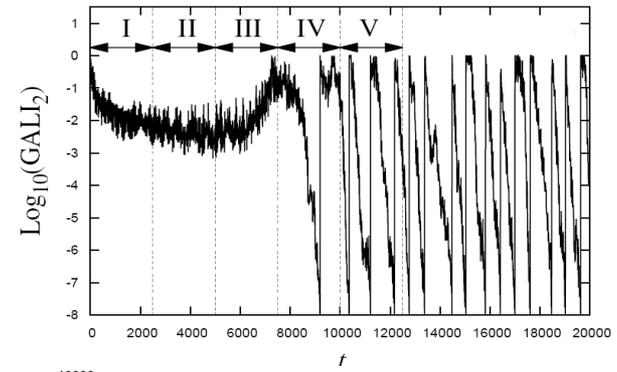
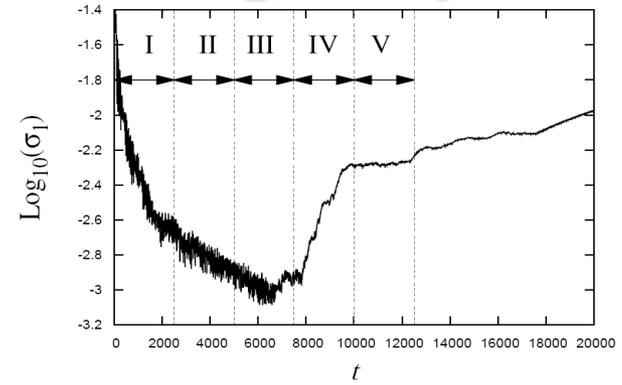
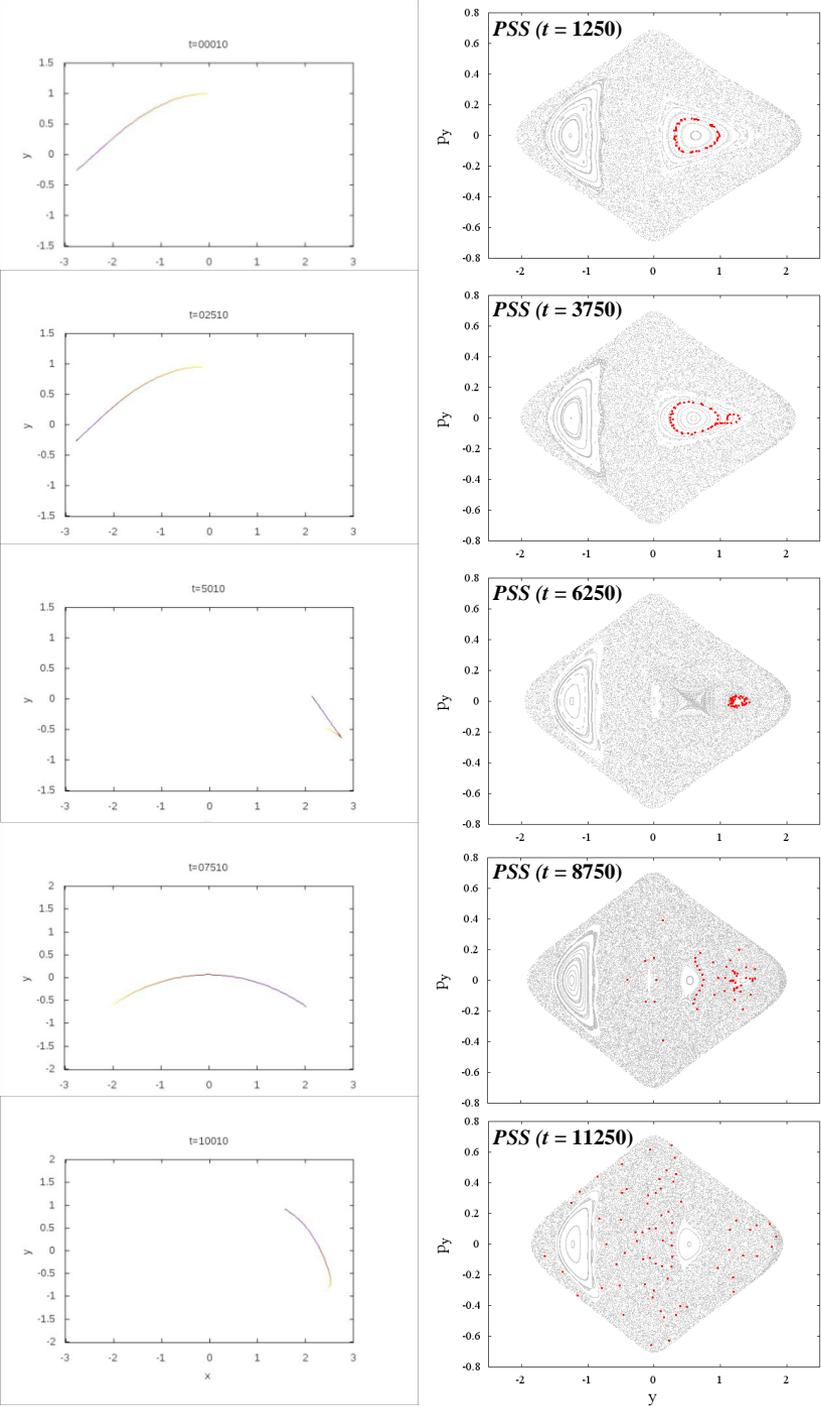
$$\rho_c = \frac{105}{32\pi} \frac{GM_B(t)}{abc}$$

n : positive integer ($n = 2$ for our model), λ : the unique positive solution of $m^2(\lambda) = 1$

Its density is:

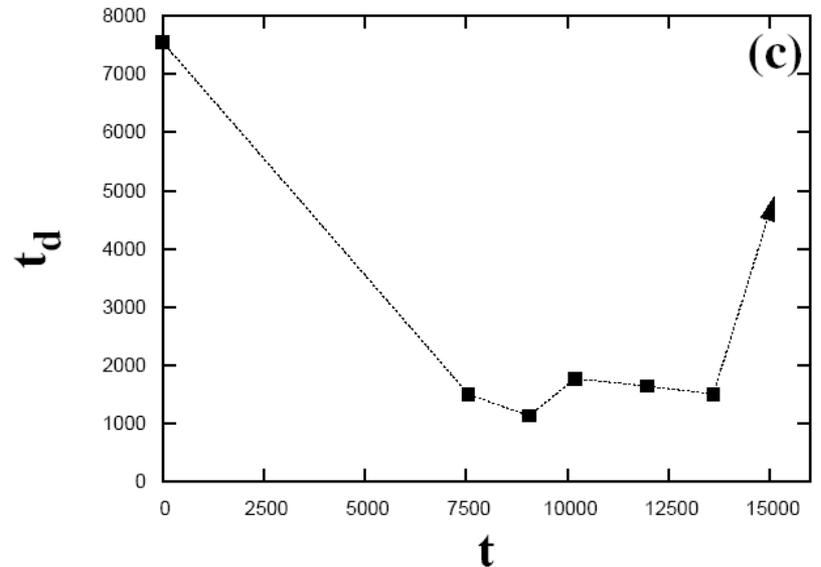
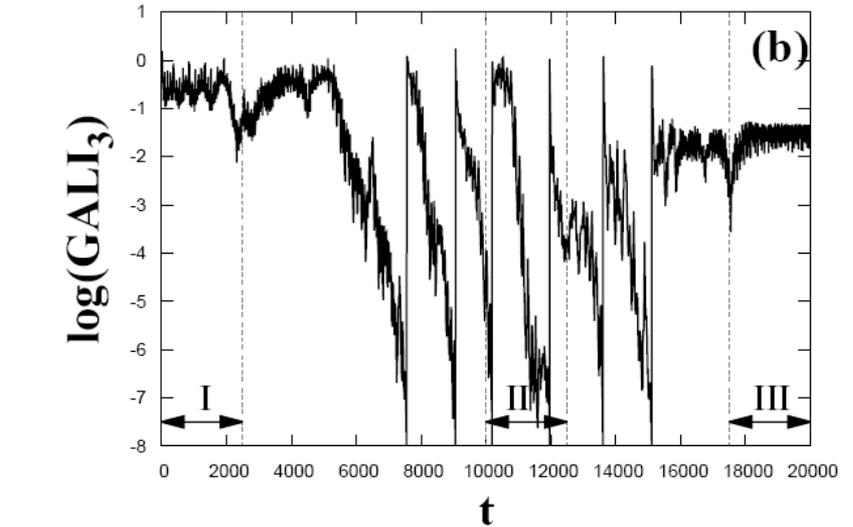
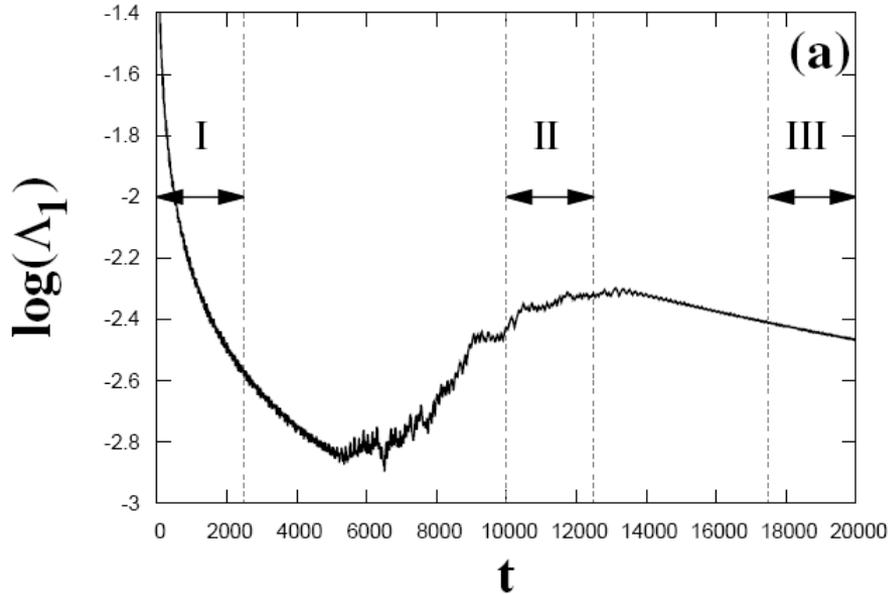
$$\rho = \begin{cases} \rho_c (1 - m^2)^n, & \text{for } m \leq 1 \\ 0, & \text{for } m > 1 \end{cases}, \text{ where } m^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}, a > b > c \text{ and } n = 2.$$

Time-dependent 2D barred galaxy model



Time-dependent 3D barred galaxy model

Interplay between chaotic and regular motion



Conclusions I

- **The Smaller ALignment Index (SALI) method a fast, efficient and easy to compute chaos indicator.**
- **Behaviour of the SALI :**
 - ✓ **2D maps: it tends to zero following completely different time rates for regular and chaotic orbits, which allows the distinction between the two cases.**
 - ✓ **Hamiltonian flows and in multidimensional maps: it goes to zero for chaotic orbits, while it tends to a positive value for ordered orbits.**

Conclusions II

- Generalizing the SALI method we define the Generalized Alignment Index of order k ($GALI_k$) as **the volume of the parallelepiped, whose edges are k unit deviation vectors. $GALI_k$ is computed as the product of the singular values of a matrix (SVD algorithm).**
- Behaviour of $GALI_k$:
 - ✓ **Chaotic motion:** **it tends exponentially to zero with exponents that involve the values of several Lyapunov exponents.**
 - ✓ **Regular motion:** **it fluctuates around non-zero values for $2 \leq k \leq s$ and goes to zero for $s < k \leq 2N$ following power-laws, with s being the dimensionality of the torus.**

Conclusions III

- **GALI_k indices :**
 - ✓ can **distinguish rapidly and with certainty between regular and chaotic motion**
 - ✓ can be used to characterize **individual orbits** as well as **"chart" chaotic and regular domains** in phase space
 - ✓ are perfectly suited for **studying the global dynamics of multidimensional systems** , as well as **of time-dependent models**
 - ✓ can identify regular **motion on low-dimensional tori**
- **SALI/GALI methods have been successfully applied to a variety of conservative dynamical systems of**
 - ✓ **Celestial Mechanics** (e.g. Széll et al., 2004, MNRAS - Soulis et al., 2008, Cel. Mech. Dyn. Astr. - Voyatzis, 2008, Astron. J. - Libert et al., 2011, MNRAS - Racoveanu, 2014, Astron. Nachr.)
 - ✓ **Galactic Dynamics** (e.g. Capuzzo-Dolcetta et al., 2007, Astroph. J. - Carpintero, 2008, MNRAS - Manos & Athanassoula, 2011, MNRAS - Carpintero et al., 2014, MNRAS)
 - ✓ **Nuclear Physics** (e.g. Macek et al., 2007, Phys. Rev. C - Stránský et al., 2007, Phys. Atom. Nucl. - Stránský et al., 2009, Phys. Rev. E - Antonopoulos et al., 2010, PRE)
 - ✓ **Statistical Physics** (e.g. Paleari & Penati, 2008, Lect. Notes Phys. - Manos & Ruffo, 2011, Trans. Theory Stat. Phys. - Christodoulidi & Efthymiopoulos, 2013, Physica D)

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